

# INNER MODELS WITH LARGE CARDINAL FEATURES USUALLY OBTAINED BY FORCING

ARTHUR W. APTER, VICTORIA GITMAN, AND JOEL DAVID HAMKINS

**ABSTRACT.** We construct a variety of inner models exhibiting features usually obtained by forcing over universes with large cardinals. For example, if there is a supercompact cardinal, then there is an inner model with a Laver indestructible supercompact cardinal. If there is a supercompact cardinal, then there is an inner model with a supercompact cardinal  $\kappa$  for which  $2^\kappa = \kappa^+$ , another for which  $2^\kappa = \kappa^{++}$  and another in which the least strongly compact cardinal is supercompact. If there is a strongly compact cardinal, then there is an inner model with a strongly compact cardinal, for which the measurable cardinals are bounded below it and another inner model  $W$  with a strongly compact cardinal  $\kappa$ , such that  $H_{\kappa^+}^V \subseteq \text{HOD}^W$ . Similar facts hold for supercompact, measurable and strongly Ramsey cardinals. If a cardinal is supercompact up to a weakly iterable cardinal, then there is an inner model of the Proper Forcing Axiom and another inner model with a supercompact cardinal in which  $\text{GCH} + V = \text{HOD}$  holds. Under the same hypothesis, there is an inner model with level by level equivalence between strong compactness and supercompactness, and indeed, another in which there is level by level inequivalence between strong compactness and supercompactness. If a cardinal is strongly compact up to a weakly iterable cardinal, then there is an inner model in which the least measurable cardinal is strongly compact. If there is a weakly iterable limit  $\delta$  of  $<\delta$ -supercompact cardinals, then there is an inner model with a proper class of Laver-indestructible supercompact cardinals. We describe three general proof methods, which can be used to prove many similar results.

## 1. INTRODUCTION

The theme of this article is to investigate the extent to which several set-theoretic properties obtainable by forcing over universes with large cardinals must also already be found in an inner model. We find this interesting in the case of supercompact and other large cardinals that seem to be beyond the current reach of the fine-structural inner model program. For example, one reason we know that the GCH is relatively consistent with many large cardinals, especially the smaller large cardinals, is that the fine-structural inner models that have been constructed for these large cardinals satisfy the GCH; another reason is that the canonical forcing of the GCH preserves all the standard large cardinals. In the case of supercompact and other very large large cardinals, we currently lack such fine-structural inner

---

*Date:* June 25, 2010 (revised October 31, 2011).

*2000 Mathematics Subject Classification.* 03E45, 03E55, 03E40.

*Key words and phrases.* Forcing, large cardinals, inner models.

The research of each of the authors has been supported in part by research grants from the CUNY Research Foundation. The third author's research has been additionally supported by research grants from the National Science Foundation and from the Simons Foundation.

models and therefore have relied on the forcing argument alone when showing relative consistency with the GCH. It seems quite natural to inquire, without insisting on fine structure, whether these cardinals nevertheless have an inner model with the GCH.

**Test Question 1.** If there is a supercompact cardinal, then must there be an inner model with a supercompact cardinal in which the GCH also holds?

**Test Question 2.** If there is a supercompact cardinal, then must there be an inner model with a supercompact cardinal  $\kappa$  such that  $2^\kappa = \kappa^+$ ?

**Test Question 3.** If there is a supercompact cardinal, then must there be an inner model with a supercompact cardinal  $\kappa$  such that  $2^\kappa > \kappa^+$ ?

These questions are addressed by our Theorems 14 and 25. We regard these test questions and the others we are about to introduce as stand-ins for their numerous variations, asking of a particular set-theoretic assertion known to be forceable over a universe with large cardinals, whether it must hold already in an inner model whenever such large cardinals exist. The questions therefore concern what we describe as the internal consistency strength of the relevant assertions, a concept we presently explain. Following ideas of Sy Friedman [Fri06], let us say that an assertion  $\varphi$  is *internally consistent* if it holds in an inner model, that is, if there is a transitive class model of ZFC, containing all the ordinals, in which  $\varphi$  is true. In this general form, an assertion of internal consistency is a second-order assertion, expressible in GBC set theory (as are our test questions); nevertheless, it turns out that many interesting affirmative instances of internal consistency are expressible in the first-order language of set theory, when the relevant inner model is a definable class, and as a result much of the analysis of internal consistency can be carried out in first-order ZFC. One may measure what we refer to as the *internal consistency strength* of an assertion  $\varphi$  by the hypothesis necessary to prove that  $\varphi$  holds in an inner model. Specifically, we say that the internal consistency strength of  $\varphi$  is bounded above by a large cardinal or other hypothesis  $\psi$ , if we can prove from  $\text{ZFC} + \psi$  that there is an inner model of  $\varphi$ ; in other words, if we can argue from the truth of  $\psi$  to the existence of an inner model of  $\varphi$ . Two statements are *internally-equiconsistent* if each of them proves the existence of an inner model of the other. It follows that the internal consistency strength of an assertion is at least as great as the ordinary consistency strength of that assertion, and the interesting phenomenon here is that internal consistency strength can sometimes exceed ordinary consistency strength. For example, although the hypothesis  $\varphi$  asserting “there is a measurable cardinal and CH fails” is equiconsistent with a measurable cardinal, because it is easily forced over any model with a measurable cardinal, nevertheless the internal consistency strength of  $\varphi$ , assuming consistency, is strictly larger than a measurable cardinal, because there are models having a measurable cardinal in which there is no inner model satisfying  $\varphi$ . For example, in the canonical model  $L[\mu]$  for a single measurable cardinal, every inner model with a measurable cardinal contains an iterate of  $L[\mu]$  and therefore agrees that CH holds. So one needs more than just a measurable cardinal in order to ensure that there is an inner model with a measurable cardinal in which CH fails.

With this sense of internal consistency strength, the reader may observe that our test questions exactly inquire about the internal consistency strength of their conclusions. For instance, Test Questions 1, 2 and 3 inquire whether the internal

consistency strength of a supercompact cardinal plus the corresponding amount of the GCH or its negation is bounded above by and hence internally-equiconsistent with the existence of a supercompact cardinal.

In several of our answers, the inner models we provide will also exhibit additional nice features; for example, in some cases we shall produce for every cardinal  $\theta$  an inner model  $W$  satisfying the desired assertion, but also having  $W^\theta \subseteq W$ . These answers therefore provide an especially strong form of internal consistency, and it would be interesting to investigate the extent to which the strong internal consistency strength of an assertion can exceed its internal consistency strength, which as we have mentioned is already known sometimes to exceed its ordinary consistency strength.

Let us continue with a few more test questions that we shall use to frame our later discussion. Forcing, of course, can also achieve large cardinal properties that we do not expect to hold in the fine-structural inner models. For example, Laver [Lav78] famously proved that after his forcing preparation, any supercompact cardinal  $\kappa$  is made (*Laver*) *indestructible*, meaning that it remains supercompact after any further  $<\kappa$ -directed closed forcing. In contrast, large cardinals are typically destructible over their fine-structural inner models (for example, see [Ham94, Theorem 1.1], and observe that the argument generalizes to many of the other fine-structural inner models; the crucial property needed is that the embeddings of the forcing extensions of the fine-structural model should lift ground model embeddings). Nevertheless, giving up the fine-structure, we may still ask for indestructibility in an inner model.

**Test Question 4.** If there is a supercompact cardinal, then must there be an inner model with an indestructible supercompact cardinal?

We answer this question in Theorem 8. For another example, recall that Baumgartner [Bau84] proved that if  $\kappa$  is a supercompact cardinal, then there is a forcing extension satisfying the Proper Forcing Axiom (PFA). We inquire whether there must in fact be an inner model satisfying the PFA:

**Test Question 5.** If there is a supercompact cardinal, then must there be an inner model satisfying the Proper Forcing Axiom?

This question is addressed by Theorem 23, using a stronger hypothesis. Next, we inquire the extent to which there must be inner models  $W$  having a very rich  $\text{HOD}^W$ . With class forcing, one can easily force  $V = \text{HOD}$  while preserving all of the most well-known large cardinal notions, and of course, one finds  $V = \text{HOD}$  in the canonical inner models of large cardinals. Must there also be such inner models for the very large large cardinals?

**Test Question 6.** If there is a supercompact cardinal, then must there be an inner model with a supercompact cardinal satisfying  $V = \text{HOD}$ ?

Since one may easily force to make any particular set  $A$  definable in a forcing extension by forcing that preserves all the usual large cardinals, another version of this question inquires:

**Test Question 7.** If there is a supercompact cardinal, then for every set  $A$ , must there be an inner model  $W$  with a supercompact cardinal such that  $A \in \text{HOD}^W$ ?

These questions are addressed by our Theorems 15 and 25. One may similarly inquire, if there is a measurable cardinal, then does every set  $A$  have an inner model with a measurable cardinal in which  $A \in \text{HOD}^W$ ? What of other large cardinal notions? What if one restricts to  $A \in H_{\kappa^+}$ ? There is an enormous family of such questions surrounding the HODs of inner models. Furthermore, apart from large cardinals, for which sets  $A$  is there an inner model  $W$  with  $A \in \text{HOD}^W$ ? There are numerous variants of this question.

More generally, whenever a feature is provably forceable in the presence of a certain large cardinal, then we ask: is there already an inner model with that feature? How robust can these inner models be?

Before continuing, we fix some terminology. Suppose  $\kappa$  is a regular cardinal. A forcing notion is  $<\kappa$ -directed closed when any directed subset of it of size less than  $\kappa$  has a lower bound. (This is what Laver in [Lav78] refers to as  $\kappa$ -directed closed.) A forcing notion is  $\leq\kappa$ -closed if any decreasing chain of length less than or equal to  $\kappa$  has a lower bound. A forcing notion is  $\leq\kappa$ -strategically closed if in the game of length  $\kappa + 1$  in which two players alternately select conditions from it to construct a descending  $(\kappa + 1)$ -sequence, with the second player playing at limit stages, the second player has a strategy that allows her always to continue playing. A forcing notion is  $<\kappa$ -strategically closed if in the game of length  $\kappa$  in which two players alternately select conditions from it to construct a descending  $\kappa$ -sequence, with the second player playing at limit stages, the second player has a strategy that allows her always to continue playing. If a poset  $\mathbb{P}$  is  $\leq\kappa$ -closed, then it is also  $\leq\kappa$ -strategically closed. If  $\lambda$  is an ordinal, then  $\text{Add}(\kappa, \lambda)$  is the standard poset for adding  $\lambda$  many Cohen subsets to  $\kappa$ . A Boolean algebra  $\mathbb{B}$  is  $(\lambda, 2)$ -distributive if the distributive law:  $\bigwedge_{\alpha < \lambda} u_{\alpha,0} \vee u_{\alpha,1} = \bigvee_{f \in 2^\lambda} \bigwedge_{\alpha < \lambda} u_{\alpha, f(\alpha)}$  holds. Equivalently, a Boolean algebra is  $(\lambda, 2)$ -distributive if every  $f : \lambda \rightarrow 2$  in the generic extension by  $\mathbb{B}$  is in the ground model. The theory  $\text{ZFC}^-$  consists of the standard ZFC axioms without the powerset axiom and with the replacement scheme replaced by the collection scheme (see [GHJ] for the significance of choosing collection over replacement). A transitive set  $M \models \text{ZFC}^-$  is a  $\kappa$ -model if  $|M| = \kappa$ ,  $\kappa \in M$  and  $M^{<\kappa} \subseteq M$ . An elementary embedding  $j : M \rightarrow N$  is said to *lift* to another elementary embedding  $j^* : M^* \rightarrow N^*$ , where  $M \subseteq M^*$  and  $N \subseteq N^*$ , if the two embeddings agree on the smaller domain, i.e.  $j^* \upharpoonright M = j$ . An elementary embedding  $j : M \rightarrow N$  having critical point  $\kappa$  is  $\kappa$ -powerset preserving if  $M$  and  $N$  have the same subsets of  $\kappa$ . A cardinal  $\kappa$  is *strongly Ramsey* if every  $A \subseteq \kappa$  is contained in a  $\kappa$ -model  $M$  for which there exists a  $\kappa$ -powerset preserving elementary embedding  $j : M \rightarrow N$ .

## 2. THREE PROOF METHODS

In order best to introduce our methods, which we view as the main contribution of this article, we shall begin with Test Question 4, which is answered by Theorem 8 below. We shall give three different arguments with this conclusion, using different proof methods (our third method will prove a slightly weaker result, because it requires a slightly stronger hypothesis). These methods are robust enough directly to answer many variants of the test questions. In Sections 4 and 5, we describe how some further modifications of the methods enable them to prove additional related results.

**Theorem 8.** *If there is a supercompact cardinal, then there is an inner model with an indestructible supercompact cardinal.*

The first proof makes use of an observation of Hamkins and Seabold involving Boolean ultrapowers (see [HS]), which is essentially encapsulated in Theorems 10 and 11.

**Definition 9.** A forcing notion  $\mathbb{P}$  is  $<\kappa$ -friendly if for every  $\gamma < \kappa$ , there is a condition  $p \in \mathbb{P}$  below which the restricted forcing  $\mathbb{P} \restriction p$  adds no subsets to  $\gamma$ .

**Theorem 10** (Hamkins, Seabold [HS]). *If  $\kappa$  is a strongly compact cardinal and  $\mathbb{P}$  is a  $<\kappa$ -friendly notion of forcing, then there is an inner model  $W$  satisfying every sentence forced by  $\mathbb{P}$  over  $V$ .*

*Proof.* The proof uses Boolean ultrapowers (see [HS] for a full account). To make this paper self-contained, we shall review the method. Suppose that  $\mathbb{B}$  is the complete Boolean algebra corresponding to the forcing notion  $\mathbb{P}$ . Let  $V^{\mathbb{B}}$  be the usual class of  $\mathbb{B}$ -names, endowed as a Boolean-valued structure by the usual recursive definition of the Boolean values  $\llbracket \varphi \rrbracket$  for every assertion  $\varphi$  in the forcing language. Now suppose that  $U \subseteq \mathbb{B}$  is an ultrafilter, not necessarily generic in any sense, and define the equivalence relation  $\sigma \in_U \tau \iff \llbracket \sigma = \tau \rrbracket \in U$ . When  $U$  is not  $V$ -generic, this relation is not the same as  $\text{val}(\sigma, U) = \text{val}(\tau, U)$ . Nevertheless, the relation  $\sigma \in_U \tau \iff \llbracket \sigma \in \tau \rrbracket \in U$  is well-defined with respect to  $\in_U$ , and we may form the quotient structure  $V^{\mathbb{B}}/U$  as the collection of (Scott's trick reduced) equivalence classes  $[\tau]_U$ . The relation  $\in_U$  is set-like, because whenever  $\sigma \in_U \tau$ , then  $\sigma$  is  $\in_U$  equivalent to a mixture of the names in the domain of  $\tau$ , and there are only set many such mixtures. One can easily establish Los' theorem that  $V^{\mathbb{B}}/U \models \varphi[[\tau]_U] \iff \llbracket \varphi(\tau) \rrbracket \in U$ . In particular, any statement  $\varphi$  that is forced by  $\mathbb{1}$  will be true in  $V^{\mathbb{B}}/U$ . Thus, since  $U$  is in  $V$ , we have produced in  $V$  a class model  $V^{\mathbb{B}}/U$  satisfying the desired theory; but there is no reason so far to suppose that this model is well-founded.

In order to find an ultrafilter  $U$  for which  $V^{\mathbb{B}}/U$  is well-founded, we shall make use of our assumption that  $\mathbb{P}$  and hence also  $\mathbb{B}$  is  $<\kappa$ -friendly for a strongly compact cardinal  $\kappa$ . Just as with classical powerset ultrapowers, the structure  $V^{\mathbb{B}}/U$  is well-founded if and only if  $U$  is countably complete (see [HS]). Next, consider any  $\theta \geq |\mathbb{B}|$  and let  $j : V \rightarrow M$  be a  $\theta$ -strong compactness embedding, so that  $j \restriction \mathbb{B} \subseteq s \in M$  for some  $s \in M$  with  $|s|^M < j(\kappa)$ . Since  $j(\mathbb{B})$  is  $<j(\kappa)$ -friendly, there is a condition  $p \in j(\mathbb{B})$  such that  $j(\mathbb{B}) \restriction p$  adds no new subsets to  $\lambda = |s|^M$ . Thus,  $j(\mathbb{B}) \restriction p$  is  $(\lambda, 2)$ -distributive in  $M$ . Applying this, it follows in  $M$  that

$$p = p \wedge 1 = p \wedge \bigwedge_{b \in s} (b \vee \neg b) = \bigwedge_{b \in s} (p \wedge b) \vee (p \wedge \neg b) = \bigvee_{f \in 2^s} \bigwedge_{b \in s} (p \wedge (\neg)^{f(b)} b),$$

where  $(\neg)^0 b = b$  and  $(\neg)^1 b = \neg b$ , and where we use distributivity to deduce the final equality. Since  $p$  is not 0, it follows that there must be some  $f$  with  $q = \bigwedge_{b \in s} p \wedge (\neg)^{f(b)} b \neq 0$ . Note that  $f(b)$  and  $f(\neg b)$  must have opposite values. Now we use  $q$  as a seed to define the ultrafilter  $U = \{a \in \mathbb{B} \mid q \leq j(a)\}$ , which is the same as  $\{a \in \mathbb{B} \mid f(j(a)) = 0\}$ . This is easily seen to be a  $\kappa$ -complete filter using the fact that  $\text{cp}(j) = \kappa$  (just as in the powerset ultrafilter cases known classically). It is an ultrafilter precisely because  $s$  covers  $j \restriction \mathbb{B}$ , so either  $f(j(a)) = 0$  or  $f(\neg j(a)) = 0$ , and so either  $a \in U$  or  $\neg a \in U$ , as desired. In summary, using this ultrafilter  $U$ ,

the structure  $V^{\mathbb{B}}/U$  is a well-founded set-like model of the desired theory. The corresponding Mostowski collapse is the desired inner model  $W$ .  $\square$

The metamathematical reader will observe that Theorem 10 is more properly described as a theorem scheme, since we defined a certain inner model, using  $\mathbb{P}$  as a parameter, and then proved of each sentence forceable by  $\mathbb{P}$  over  $V$ , that this sentence also holds in the inner model. By Tarski's theorem on the non-definability of truth, it does not seem possible to state the conclusion of Theorem 10 in a single first order statement. Several similar theorems in this article will also be theorem schemes.

The following account of the Boolean ultrapower may be somewhat more illuminating.

**Theorem 11 ([HS]).** *If  $\kappa$  is strongly compact and  $\mathbb{P}$  is  $<\kappa$ -friendly, then there is an elementary embedding  $j : V \rightarrow \bar{V}$  into an inner model  $\bar{V}$  and a  $\bar{V}$ -generic filter  $G \subseteq j(\mathbb{P})$  with  $G \in V$ . In particular,  $W = \bar{V}[G]$  fulfills Theorem 10.*

*Proof.* This is actually what is going on in the Boolean quotient. We may define the canonical predicate for the ground model of  $V^{\mathbb{B}}$  by  $\llbracket \tau \in \check{V} \rrbracket = \bigvee_{x \in V} \llbracket \tau = \check{x} \rrbracket$ , and let  $\bar{V} = \{[\tau]_U \mid \llbracket \tau \in \check{V} \rrbracket \in U\}$ , which is actually the same as  $\{[\tau]_U \mid \llbracket \tau \in \check{V} \rrbracket = \mathbb{1}\}$ . An easy induction on formulas shows that the map  $j : x \mapsto [\check{x}]_U$  is an elementary embedding  $j : V \rightarrow \bar{V}$ , and this is the map known as the Boolean ultrapower. As was observed in [HS], the critical point of  $j$  is the cardinality of the smallest maximal antichain not met by the  $U$ , which in this case must be at least  $\kappa$  since  $U$  is  $\kappa$ -complete. If  $\dot{G}$  is the (usual) canonical name for the generic filter, then  $\llbracket \dot{G} \text{ is } \check{V}\text{-generic for } \mathbb{B} \rrbracket = 1$ , and so the corresponding equivalence class  $G = [\dot{G}]_U$  is  $\bar{V}$ -generic for  $[\mathbb{B}]_U = j(\mathbb{B})$ . Since these embeddings and equivalence classes all exist in  $V$ , we have the entire Boolean ultrapower

$$j : V \rightarrow \bar{V} \subseteq \bar{V}[G]$$

existing in  $V$ , as desired. The structure  $\bar{V}[G]$  is isomorphic to the quotient  $V^{\mathbb{B}}/U$  by the map associating  $[\tau]_U = [\text{val}(\check{\tau}, \dot{G})]_U$  in  $V^{\mathbb{B}}/U$  with  $\text{val}([\check{\tau}]_U, G)$  in  $\bar{V}[G]$ .  $\square$

Certain instances of this phenomenon are already well known. For example, consider Prikry forcing with respect to a normal measure  $\mu$  on a measurable cardinal  $\kappa$ , which is  $<\kappa$ -friendly because it adds no bounded subsets to  $\kappa$ . If  $V \rightarrow M_1 \rightarrow M_2 \rightarrow \dots$  is the usual iteration of  $\mu$ , with a direct limit to  $j_\omega : V \rightarrow M_\omega$ , then the critical sequence  $\kappa_0, \kappa_1, \kappa_2, \dots$  is well known to be  $M_\omega$ -generic for the corresponding Prikry forcing at  $j_\omega(\kappa)$  using  $j_\omega(\mu)$ . This is precisely the situation occurring in Theorem 11, where we have an embedding  $j : V \rightarrow \bar{V}$  and a  $\bar{V}$ -generic filter  $G \subseteq j(\mathbb{P})$  all inside  $V$ . Thus, Theorem 11 generalizes this classical aspect about Prikry forcing to all friendly forcing under the stronger assumption of strong compactness.

We now derive Theorem 8 as a corollary.

*Proof of Theorem 8.* We shall apply Theorem 10 by finding a  $<\kappa$ -friendly version of the Laver preparation. The original Laver preparation of [Lav78] is not friendly, because there are many stages  $\gamma < \kappa$  at which it definitely adds, for example, a Cohen subset to  $\gamma$ . But a relatively simple modification will make it  $<\kappa$ -friendly. Suppose that  $\ell : \kappa \rightarrow V_\kappa$  is a Laver function. It follows easily that the restriction  $\ell \upharpoonright (\gamma, \kappa)$  to any final segment  $(\gamma, \kappa)$  of  $\kappa$  is also a Laver function, and the corresponding Laver preparation  $\mathbb{P}_{\ell \upharpoonright (\gamma, \kappa)}$  is  $\leq \gamma$ -closed, hence adding no new subsets to  $\gamma$ , while

still forcing indestructibility for  $\kappa$ . Let  $\mathbb{P} = \oplus \{\mathbb{P}_{\ell \upharpoonright (\gamma, \kappa)} \mid \gamma < \kappa\}$  be the lottery sum of all these various preparations<sup>1</sup>, so that the generic filter in effect selects a single  $\gamma$  and then forces with  $\mathbb{P}_{\ell \upharpoonright (\gamma, \kappa)}$ . This poset is  $<\kappa$ -friendly, since a condition could opt in the lottery to use a preparation with  $\gamma$  as large below  $\kappa$  as desired. The point is that the Laver preparation works fine for indestructibility even if we allow it to delay the start of the forcing as long as desired, and such a modification makes it  $<\kappa$ -friendly. So Theorem 10 applies, and Theorem 8 now follows as a corollary.  $\square$

After realizing that Theorem 8 could be proved via Boolean ultrapowers, we searched for a direct proof. We arrived at the following stronger result, which produces more robust inner models  $W$ , satisfying a closure condition  $W^\theta \subseteq W$ .

**Theorem 12.** *If there is a supercompact cardinal, then for every cardinal  $\theta$  there is an inner model  $W$  with an indestructible supercompact cardinal, such that  $W^\theta \subseteq W$ .*

*Proof.* Suppose that  $\kappa$  is supercompact. By a result of Solovay [Sol74], the SCH holds above  $\kappa$ , and so if  $\theta$  is any singular strong limit cardinal of cofinality at least  $\kappa$ , then  $2^{\theta^{<\kappa}} = \theta^+$ . Consider any such  $\theta$  as large as desired above  $\kappa$ , and let  $j : V \rightarrow M$  be a  $\theta$ -supercompactness embedding, the ultrapower by a normal fine measure on  $P_\kappa(\theta)$ . Thus,  $\theta < j(\kappa)$  and  $M^\theta \subseteq M$ . By elementarity,  $j(\kappa)$  is supercompact in  $M$ . Let  $\mathbb{P}$  be the Laver preparation of  $j(\kappa)$  in  $M$ , with nontrivial forcing only in the interval  $(\theta, j(\kappa))$ . That is, we put off the start of the Laver preparation until beyond  $\theta$ , and this is exactly what corresponds to the use of friendliness in the earlier proof. Notice that  $\mathbb{P}$  is  $\leq \theta$ -closed in  $M$ , and therefore also  $\leq \theta$ -closed in  $V$ . But also,  $\mathbb{P}$  has size  $j(\kappa)$  in  $M$ , and has at most  $j(2^\kappa)$  many dense subsets in  $M$ . Observe in  $V$  that  $|j(2^\kappa)| \leq (2^\kappa)^{\theta^{<\kappa}} \leq (2^{\theta^{<\kappa}})^{\theta^{<\kappa}} = 2^{\theta^{<\kappa}} = \theta^+$ . In  $V$  we may therefore enumerate the dense subsets of  $\mathbb{P}$  in  $M$  in a  $\theta^+$  sequence, and using the fact that  $\mathbb{P}$  is  $\leq \theta$ -closed, diagonalize to meet them all. So there is in  $V$  an  $M$ -generic filter  $G \subseteq \mathbb{P}$ . Thus,  $M[G]$  is an inner model of  $V$ , in which  $j(\kappa)$  is an indestructible supercompact cardinal. Since  $M^\theta \subseteq M$ , it follows that  $M[G]$  contains all  $\theta$ -sequences of ordinals in  $V$ , and so also  $M[G]^\theta \subseteq M[G]$ . So  $W = M[G]$  is as desired.  $\square$

This second method of proof can be generalized to the following, where we define that  $\mathbb{P}$  is  $<\kappa$ -superfriendly, if for every  $\gamma < \kappa$  there is a condition  $p \in \mathbb{P}$  such that  $\mathbb{P} \upharpoonright p$  is  $\leq \gamma$ -strategically closed. It was the superfriendliness of the Laver preparation that figured in the proof of Theorem 12 and the proof generalizes in a straightforward way to obtain the theorem below.

**Theorem 13.** *If  $\kappa$  is supercompact and  $\mathbb{P}$  is  $<\kappa$ -superfriendly, then for every  $\theta$  there is an inner model  $W$  satisfying every statement forced by  $\mathbb{P}$  over  $V$  and for which  $W^\theta \subseteq W$ .*

We can now solve several more of the test questions as corollaries.

**Theorem 14.** *If there is a supercompact cardinal, then there is an inner model with an indestructible supercompact cardinal  $\kappa$  such that  $2^\kappa = \kappa^+$ , and another inner model with an indestructible supercompact cardinal  $\kappa$  such that  $2^\kappa = \kappa^{++}$ . Thus, the answers to Test Questions 2, 3 and 4 are yes. Indeed, for any cardinal  $\theta$ , such inner models  $W$  can be found for which also  $W^\theta \subseteq W$ .*

<sup>1</sup>If  $\mathcal{A} = \{\mathbb{P}_i \mid i \in I\}$ , then the lottery sum  $\oplus \mathcal{A}$  is the partial order with underlying set  $\{(\mathbb{P}, p) \mid \mathbb{P} \in \mathcal{A} \text{ and } p \in \mathbb{P}\} \cup \{\mathbf{1}\}$ , ordered by  $\langle \mathbb{P}, p \rangle \leq \langle \mathbb{Q}, q \rangle$  if and only if  $\mathbb{P} = \mathbb{Q}$  and  $p \leq q$ , with  $\mathbf{1}$  above everything. The lottery preparation of [Ham00] employs long iterations of such sums.

*Proof.* Let  $\mathbb{P}$  be the  $<\kappa$ -friendly version of the Laver preparation used in Theorem 8, which is easily seen to be  $<\kappa$ -superfriendly, and let  $\dot{\mathbb{Q}} = \text{Add}(\kappa^+, 1)$  be the subsequent forcing to ensure  $2^\kappa = \kappa^+$ . The combination  $\mathbb{P} * \dot{\mathbb{Q}}$  remains  $<\kappa$ -superfriendly, forces  $2^\kappa = \kappa^+$  and preserves the indestructible supercompactness of  $\kappa$ . Thus, by either Theorem 10 or 13, there is an inner model satisfying this theory. Similarly, if  $\dot{\mathbb{R}} = \text{Add}(\kappa, \kappa^{++})$ , then  $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$  is  $<\kappa$ -superfriendly, preserves the indestructible supercompactness of  $\kappa$  and forces  $2^\kappa = \kappa^{++}$ , so again there is an inner model of the desired theory. The method of Theorem 13 will ensure in each case, for any desired cardinal  $\theta$ , that the inner model  $W$  satisfies  $W^\theta \subseteq W$ .  $\square$

The proof admits myriad alternatives. For example, we could have just as easily forced  $2^\kappa = \kappa^{+++}$ , or GCH on a long block of cardinals at  $\kappa$  and above, or failures of this, in any definable pattern above  $\kappa$ . If  $\dot{\mathbb{Q}}$  is any  $<\kappa$ -directed closed forcing, to be performed after the (superfriendly) Laver preparation, then the combination  $\mathbb{P} * \dot{\mathbb{Q}}$  is  $<\kappa$ -superfriendly and preserves the indestructible supercompactness of  $\kappa$ . Thus, any statement forced by  $\mathbb{P} * \dot{\mathbb{Q}}$ , using any parameter in  $V_\kappa$ , will be true in the inner models  $W$  arising in Theorems 10 and 13. See also Theorem 15 for an application of this method.

We now apply these methods to the family of questions surrounding Test Questions 6 and 7. The following theorem answers Test Question 7 and several of its variants, but not Test Question 6. With our third proof method in a later section, we will deduce the full conclusion of Test Question 6 using a slightly stronger hypothesis.

**Theorem 15.**

- (1) *If  $\kappa$  is strongly compact, then there is an inner model  $W$  with a strongly compact cardinal, such that  $H_{\kappa^+}^V \subseteq \text{HOD}^W$ . If the GCH holds below  $\kappa$ , then for any  $A \in H_{\kappa^+}^V$ , one can arrange that  $A$  is definable in  $W$  without parameters.*
- (2) *If  $\kappa$  is measurable and  $2^\kappa = \kappa^+$ , then there is an inner model  $W$  with a measurable cardinal, such that  $H_{\kappa^+}^V \subseteq \text{HOD}^W$ . If the GCH holds below  $\kappa$ , then for any  $A \in H_{\kappa^+}^V$ , one can arrange that  $A$  is definable in  $W$  without parameters.*
- (3) *If  $\kappa$  is supercompact, then for every cardinal  $\theta$  and every set  $A \in H_{\theta^+}^V$ , there is an inner model  $W$  with a supercompact cardinal in which  $A \in \text{HOD}^W$  and  $W^\theta \subseteq W$ . If the GCH holds below  $\kappa$ , then one can arrange that  $A$  is definable in  $W$  without parameters.*

In particular, by Theorem 15(3), the answer to Test Question 7 is yes.

*Proof.* For Statement (1), we use the Boolean ultrapower method of Theorems 10 and 11. For  $\gamma < \kappa$ , let  $\mathbb{Q}_\gamma$  be the poset that codes  $P(\gamma)$  into the GCH pattern on a block of cardinals above  $\gamma$ , using  $\leq \gamma$ -closed forcing. Let  $\mathbb{P}$  be the lottery sum  $\oplus \{\mathbb{Q}_\gamma \mid \gamma < \kappa\}$ , which is  $<\kappa$ -superfriendly and therefore  $<\kappa$ -friendly. Since each  $\mathbb{Q}_\gamma$  is small relative to  $\kappa$ , it follows by the results of [LS67] that  $\kappa$  remains strongly compact after forcing with  $\mathbb{P}$ . By Theorem 11, there is an embedding  $j : V \rightarrow \bar{V}$  into an inner model  $\bar{V}$  with  $\kappa \leq \text{cp}(j)$ , and in  $V$ , there is a  $\bar{V}$ -generic filter  $G \subseteq j(\mathbb{P})$ . In particular,  $V_\kappa = \bar{V}_\kappa$ . Consequently, the ordinal  $\gamma$  selected by  $G$  in the lottery  $j(\mathbb{P})$  must be at least  $\kappa$ . Thus,  $G$  is coding  $P(\gamma)$  and hence also  $P(\kappa)$  into the continuum function in some interval between  $\kappa$  and  $j(\kappa)$ . So in the inner model



$W = \overline{V}[G]$ , we have a strongly compact cardinal  $j(\kappa)$ , as well as  $P(\kappa)^V \subseteq \text{HOD}^W$  and hence  $H_{\kappa^+}^V \subseteq \text{HOD}^W$ , as desired.

For the second part of Statement (1), in the case that the GCH holds below  $\kappa$  in  $V$ , consider any  $A \in H_{\kappa^+}$ . Let  $\hat{A} \subseteq \kappa$  be a subset of  $\kappa$  coding  $A$  in some canonical way. Let  $\mathbb{P}$  be the forcing as in the previous paragraph, but modified so that the lottery also may choose the order in which the sets in  $P(\gamma)$  are coded. First, we argue that we may assume that  $G$  opts for a poset in  $j(\mathbb{P})$  that begins coding at  $\gamma$ , which is the successor of a cardinal of cofinality  $\kappa$ , and that  $\hat{A}$  is the first set to be coded. Note that the statement  $\varphi$  that  $G$  makes such a choice is expressible as an assertion in the forcing language, and so, following the proof of Theorem 10, it suffices to obtain an ultrafilter  $U$  for the complete Boolean algebra  $\mathbb{B}$  corresponding to the poset  $\mathbb{P}$  containing the Boolean value of  $\varphi$ . Fixing a strong compactness embedding  $h$ , we obtain  $U$  precisely as in the proof of Theorem 10, only making sure that the condition  $p \in h(\mathbb{B})$  chosen to witness the friendliness of  $h(\mathbb{B})$  for  $\lambda = |s|^M < j(\kappa)$ , where  $j \restriction \mathbb{B} \subseteq s$ , forces the statement  $\varphi$  with  $h$  applied to the parameters. Now observe that in  $W = \overline{V}[G]$ , the cardinal  $\gamma$  is definable as the cardinal up to which the GCH holds, since in  $\overline{V}$  the GCH holds below  $j(\kappa)$  by elementarity. It follows that  $\kappa$  is definable as the cofinality of the predecessor of  $\gamma$ , and so  $\hat{A}$  and hence  $A$  are definable in  $W$  without parameters. As in the previous paragraph, we also have a strongly compact cardinal in  $W$  and  $H_{\theta^+}^V \subseteq W$ .

For Statement (2), suppose that  $\kappa$  is measurable and  $2^\kappa = \kappa^+$ . We follow the method of Theorem 12. Let  $j : V \rightarrow M$  be the ultrapower by any normal measure on  $\kappa$ . Let  $\mathbb{P}$  be the forcing used to prove the second part of Statement (1), which by lottery selects some  $\gamma < \kappa$  and an enumeration of  $P(\gamma)$ , which is then coded into the GCH pattern above  $\gamma$ . In the forcing  $j(\mathbb{P})$ , consider a condition  $p$  that opts to code  $P(\kappa)$ . Thus,  $j(\mathbb{P}) \restriction p$  is  $\leq \kappa$ -closed and has size less than  $j(\kappa)$ . Since  $2^\kappa = \kappa^+$ , the number of subsets of  $j(\mathbb{P})$  in  $M$ , counted in  $V$ , is bounded by  $|j(2^\kappa)|^V \leq (2^\kappa)^\kappa = \kappa^+$ . Thus, by diagonalization, we may construct in  $V$  an  $M$ -generic filter  $G \subseteq j(\mathbb{P})$  below  $p$ . Let  $W = M[G]$ . By the results of [LS67], the cardinal  $j(\kappa)$  remains measurable in  $W$ , since below  $p$  the forcing was small relative to  $j(\kappa)$ . In addition, every set in  $P(\kappa)^V = P(\kappa)^M$  is coded into the continuum function of  $W$ , so  $H_{\kappa^+}^V \subseteq \text{HOD}^W$ , as desired. If the GCH holds below  $\kappa$ , then it holds below  $j(\kappa)$  in  $M$ , and so we can define  $\kappa$  in  $W$  as the cardinal up to which the GCH holds, and hence define the first set that is coded without parameters.

For Statement (3), where  $\kappa$  is supercompact, we may use the same argument as in Statements (1) and (2), but employing the method of Theorem 12. Let  $j : V \rightarrow M$  be a  $\theta$ -supercompactness embedding, so that in particular  $M^\theta \subseteq M$  and  $\theta < j(\kappa)$ . Let  $\mathbb{P}$  be the forcing from Statement (2), and in the forcing  $j(\mathbb{P})$ , consider a condition  $p$  that opts to code  $P(\theta)$ . Note that  $j(\mathbb{P}) \restriction p$  is  $\leq \theta$ -closed. By the proof of Theorem 12, we know that  $V$  has an  $M$ -generic filter  $G$  containing  $p$ . Let  $W = M[G]$ , and note that  $j(\kappa)$  remains supercompact in  $W$  by [LS67]. Since  $M^\theta \subseteq M$ , it follows that  $W^\theta \subseteq W$  and  $H_{\theta^+}^V = H_{\theta^+}^M \subseteq \text{HOD}^W$ , as desired. If the GCH holds below  $\kappa$ , then it holds below  $j(\kappa)$  in  $M$ , and so we can define  $\theta$  in  $W$  as the cardinal up to which the GCH holds and hence define the first set that is coded without parameters.  $\square$

Let us highlight the consequences of this theorem with a quick example. Namely, suppose that  $\kappa$  is strongly compact in  $V$  and the GCH holds. Both of these statements remain true in the forcing extension  $V[c]$  obtained by adding a single  $V$ -generic Cohen real  $c$ . Since this forcing is almost homogeneous, we know  $c$  is not in  $\text{HOD}^{V[c]}$ . Nevertheless, by Theorem 15, there are inner models  $W \subseteq V[c]$  such that  $H_{\kappa^+}^V \subseteq \text{HOD}^W$ , with a strongly compact cardinal, in which  $c$  is definable without parameters!

Our first two proof methods were able to answer several of the test questions with the provably optimal hypothesis and, moreover, while also producing inner models with some nice features, such as  $W^\theta \subseteq W$  for any desired  $\theta$ . Nevertheless, and perhaps as a consequence, these methods seem unable to produce inner models in which the full GCH holds, say, if the CH fails in  $V$ , because the resulting inner models for those methods will agree with  $V$  up to  $V_\kappa$  and beyond, where  $\kappa$  is the initial supercompact cardinal. Similarly, neither method seems able to produce an inner model in which the PFA holds, since the only known forcing to attain this—a long countable support iteration of proper forcing—adds Cohen reals unboundedly often and is therefore highly non-friendly. Furthermore, the methods seem not easily to accommodate class forcing, and allow us only to put particular sets  $A$  into  $\text{HOD}^W$  for an inner model  $W$ , without having  $W$  fully satisfy  $V = \text{HOD}$ . Therefore, these methods seem unable to answer Test Questions 1, 5 and 6. (With our third proof method, we shall give partial answers to these questions in Theorems 23 and 25, by using a stronger hypothesis.) Another unusual feature of our first two methods, as used in Theorems 8, 12 and 14, is that it is not the same supercompact cardinal  $\kappa$  that is found to be supercompact in the desired inner model. Rather, it is in each case the ordinal  $j(\kappa)$  that is found to be supercompact (and indestructible or with fragments or failures of the GCH) in an inner model. A modified version of Test Question 4 could ask, after all, whether every supercompact cardinal  $\kappa$  is itself indestructibly supercompact in an inner model. For precisely this question, we don't know, but if  $\kappa$  is supercompact up to a weakly iterable cardinal above  $\kappa$ , then the answer is yes by Theorem 22. (See Section 3 for the definition of weakly iterable cardinal.)

So let us now turn to the third method of proof, which will address these concerns, at the price of an additional large cardinal hypothesis. We shall use this method to produce an inner model with a supercompact cardinal and the full GCH, an inner model of the PFA and an inner model where  $\kappa$  itself is indestructibly supercompact, among other possibilities. The method is very similar to the methods introduced and fruitfully applied by Sy Friedman [Fri06] and by Sy Friedman and Natasha Dobrinen [DF08], [DF10], where they construct class generic filters in  $V$  over an inner model  $W$ . Also Ralf Schindler, in a personal communication with the third author, used a version of the method to provide an answer to Test Question 5, observing that if there is a supercompact cardinal with a measurable cardinal above it, then there is an inner model of the PFA.

**Theorem 16** (Schindler). *If there is a supercompact cardinal with a measurable cardinal above it, then there is an inner model of the PFA.*

The basic idea is that if  $\kappa$  is supercompact and  $\kappa < \delta$  for some measurable cardinal  $\delta$ , then one finds a countable elementary substructure  $X \prec V_\theta$ , with  $\delta \ll \theta$ , whose Mostowski collapse is a countable iterable structure with a supercompact cardinal  $\kappa_0$  below a measurable cardinal  $\delta_0$ . By iterating the measurable cardinal

$\delta_0$  of this structure out of the universe, one arrives at a full inner model  $M$ , and because  $\kappa_0$  was below the critical point of the iteration, which is  $\delta_0$ , it follows that both  $\kappa_0$  and even  $P(\kappa_0)^M$  are countable in  $V$ . Thus, by the usual diagonalization in  $V$ , there is an  $M$ -generic filter  $G$  for the Baumgartner PFA forcing (or whatever other forcing was desired), and so  $M[G]$  is the desired inner model. This method generalizes to *any* forcing notion below a measurable cardinal.

In the subsequent sections of this article, we shall elaborate on the details of this argument, while also explaining how to reduce the hypothesis from a measurable cardinal above the supercompact cardinal to merely a weakly iterable cardinal above. The construction encounters a few complications in the class-length forcing iterations, since (unlike the argument above) these iterations will be stretched to proper class size during the iteration, and so one cannot quite so easily produce the desired  $M$ -generic filter. Nevertheless, the new method remains fundamentally similar to the argument we described in the previous paragraph. Finally, we shall give several additional applications of the method.

### 3. ITERABLE STRUCTURES

We now develop some basic facts about iterable structures, which shall be sufficient to carry out the third proof method. In particular, we shall review the fact that any structure elementarily embedding into an iterable structure is itself iterable, and for a special class of forcing required in later arguments, we shall give sufficient conditions for a forcing extension of a countable iterable structure to remain iterable.

Consider structures of the form  $\langle M, \delta, U \rangle$  where  $M \models \text{ZFC}^-$  is transitive,  $\delta$  is a cardinal in  $M$ , and  $U \subseteq \mathcal{P}(\delta)^M$ . The set  $U$  is an  $M$ -ultrafilter, if  $\langle M, \delta, U \rangle \models$  “ $U$  is a normal ultrafilter”. An  $M$ -ultrafilter  $U$  is *weakly amenable* if  $U \cap A \in M$  for every set  $A$  of size  $\delta$  in  $M$ . By using only the equivalence classes of functions in  $M$ , an  $M$ -ultrafilter suffices for the usual ultrapower construction. It is easy to see that  $U$  is weakly amenable exactly when  $M$  and the ultrapower of  $M$  by  $U$  have the same subsets of  $\delta$ , that is the ultrapower embedding is  $\delta$ -powerset preserving. In this case, it turns out that one can define the iterated ultrapowers of  $M$  by  $U$  to any desired ordinal length. We say that  $\langle M, \delta, U \rangle$  is *iterable* if  $U$  is a weakly amenable  $M$ -ultrafilter and all of these resulting iterated ultrapowers are well-founded.

**Definition 17.** A cardinal  $\delta$  is *weakly iterable* if there is an iterable structure  $\langle M, \delta, U \rangle$  containing  $V_\delta$  as an element.

It is easy to see that measurable cardinals are weakly iterable. Ramsey cardinals also are weakly iterable, since if  $\delta$  is Ramsey, every  $A \subseteq \delta$  is an element of an iterable structure  $\langle M, \delta, U \rangle$  (see [Mit79]) and so there is an iterable structure containing a subset of  $\delta$  that Mostowski collapses to  $V_\delta$ . On the other hand, a weakly iterable cardinal need not even be regular. For example, every measurable cardinal remains weakly iterable after Prikry forcing, because the ground model iterable structures still exist. More generally, we claim that the least weakly iterable cardinal must have cofinality  $\omega$ . To see this, suppose that  $\delta$  is a weakly iterable cardinal of uncountable cofinality with the iterable structure  $\langle M, \delta, U \rangle$ . We shall argue that there is a smaller weakly iterable cardinal of cofinality  $\omega$ . Choose  $X_0 \prec M$  for some countable  $X_0$  containing  $\delta$ , and let  $\gamma_0 = \sup(X_0 \cap \delta)$ . Inductively define  $X_{n+1} \prec M$  with  $\gamma_{n+1} = \sup(X_{n+1} \cap \delta) < \delta$  satisfying  $V_{\gamma_{n+1}} \subseteq X_{n+1}$  and  $|X_{n+1}| < \delta$ . This

is possible since  $\delta$  is inaccessible in  $M$ , so the witnesses we need to add to  $X_{n+1}$  below  $\delta$  will be bounded below  $\delta$ , even if  $\delta$  may be singular in  $V$ . Observe that if  $X_\omega = \bigcup_{n \in \omega} X_n$  and  $\langle N, \gamma, W \rangle$  is the collapse of the structure  $\langle X_\omega, \delta, U \cap X_\omega \rangle$ , then  $\delta$  collapses to  $\gamma = \sup_n \gamma_n$  and so  $V_\gamma \in N$ . The iterability of  $\langle N, \gamma, W \rangle$  will follow from Lemma 18 below, completing the argument that  $\gamma$  is a weakly iterable cardinal of cofinality  $\omega$  below  $\delta$ .

If  $\delta$  is weakly iterable with the iterable structure  $\langle M, \delta, U \rangle$ , then  $\delta$  is at least ineffable in  $M$  and therefore, the existence of weakly iterable cardinals carries at least this large cardinal strength (see [Git11]). In fact, weakly iterable cardinals cannot exist in  $L$  (see [GW11]), but it follows from [Wel04] that they are weaker than an  $\omega_1$ -Erdős cardinal. Note that the inaccessibility of  $\delta$  in the domain of the iterable structure witnessing its weak iterability implies that it is a  $\beth$ -fixed point and  $V_\delta \models \text{ZFC}$ , by the absoluteness of satisfaction.

**Lemma 18.** *Suppose  $\langle M, \delta, U \rangle$  is iterable. Suppose further that  $\langle N, \gamma, W \rangle$  is a structure for which there exists an elementary embedding  $\rho : N \rightarrow M$  in the language  $\{\in\}$  with  $\rho(\gamma) = \delta$  and the additional property that whenever  $x \in N$  is such that  $x \subseteq W$ , then  $\rho(x) \subseteq U$ . Then  $\langle N, \gamma, W \rangle$  is iterable as well.*

*Proof.* This is a standard idea. We shall demonstrate the iterability of  $\langle N, \gamma, W \rangle$  by elementarily embedding the iterated ultrapowers of  $N$  by  $W$  into the iterated ultrapowers of  $M$  by  $U$ . Let  $\{j_{\xi\gamma} : M_\xi \rightarrow M_\gamma \mid \xi < \gamma \in \text{Ord}\}$  be the directed system of iterated ultrapowers of  $M = M_0$  with the associated sequence of ultrafilters  $\{U_\xi \mid \xi \in \text{Ord}\}$ , where  $U_0 = U$ . Also, let  $\{h_{\xi\gamma} : N_\xi \rightarrow N_\gamma \mid \xi < \gamma < \alpha\}$  be the not necessarily well-founded directed system of iterated ultrapowers of  $N = N_0$  with the associated sequence of ultrafilters  $\{W_\xi \mid \xi \in \text{Ord}\}$ , where  $W_0 = W$ . Let  $\{W_0^i : i \in I\}$  be any enumeration of all subsets of  $W_0$  that are elements of  $N_0$ , and define  $W_\xi^i = h_{0\xi}(W_0^i)$ . By induction on  $\xi$ , it is easy to see that  $W_\xi = \bigcup_{i \in I} W_\xi^i$ . We shall show that the following diagram commutes:

$$\begin{array}{ccccccc}
 M_0 & \xrightarrow{j_{01}} & M_1 & \xrightarrow{j_{12}} & M_2 & \xrightarrow{j_{23}} & \dots \xrightarrow{j_{\xi\xi+1}} M_{\xi+1} \xrightarrow{j_{\xi+1\xi+2}} \dots \\
 \uparrow \rho_0 & & \uparrow \rho_1 & & \uparrow \rho_2 & & \uparrow \rho_{\xi+1} \\
 N_0 & \xrightarrow{h_{01}} & N_1 & \xrightarrow{h_{12}} & N_2 & \xrightarrow{h_{23}} & \dots \xrightarrow{h_{\xi\xi+1}} N_{\xi+1} \xrightarrow{h_{\xi+1\xi+2}} \dots
 \end{array}$$

where

- (1)  $\rho_{\xi+1}([f]_{W_\xi}) = [\rho_\xi(f)]_{U_\xi}$ ,
- (2) if  $\lambda$  is a limit ordinal and  $t$  is a thread in the direct limit  $N_\lambda$  with domain  $[\beta, \lambda)$ , then  $\rho_\lambda(t) = j_{\beta\lambda}(\rho_\beta(t(\beta)))$ , and
- (3)  $\rho_\xi(W_\xi^i) \subseteq U_\xi$ .

We shall argue that the  $\rho_\xi$  exist by induction on  $\xi$ . Let  $\rho_0 = \rho$ , and note that  $\rho_0$  satisfies condition (3) by hypothesis. Suppose inductively that  $\rho_\xi : N_\xi \rightarrow M_\xi$  is an elementary embedding satisfying condition (3). Define  $\rho_{\xi+1}$  as in condition (1) above. Using that  $\rho_\xi(W_\xi^i) \subseteq U_\xi$  by the inductive assumption, and  $W_\xi = \bigcup W_\xi^i$ , it follows, in particular, that whenever  $A \in W_\xi$ , then  $\rho_\xi(A) \in U_\xi$ . It follows that  $\rho_{\xi+1}$  is a well-defined map and an elementary embedding. The commutativity of the diagram is also clear. It remains to verify that  $\rho_{\xi+1}(W_{\xi+1}^i) \subseteq U_{\xi+1}$ . Recall that

$$W_{\xi+1}^i = h_{\xi\xi+1}(W_\xi^i) = [c_{W_\xi^i}]_{W_\xi}.$$

Let  $\rho_\xi(W_\xi^i) = v$ . Then by the inductive assumption, we have  $v \subseteq U_\xi$ . Thus,

$$\rho_{\xi+1}(W_{\xi+1}^i) = [c_v]_{U_\xi} = j_{\xi\xi+1}(v) \subseteq U_{\xi+1}.$$

The last relation follows since  $v \subseteq U_\xi$ . This completes the inductive step. The limit case also follows easily.  $\square$

Note that if  $\rho$  is an elementary embedding in the language with the predicate for the ultrafilter, then the additional hypothesis of Lemma 18 follows for free. This is how Lemma 18 will be used in most applications below.

In the next section, we shall build inner models by iterating out these countable iterable structures and forcing over the limit model inside the universe, just as we explained in the proof sketch for Theorem 16. In other arguments, however, the desired forcing will be stretched to proper class length, and so we shall proceed instead by first forcing over the countable structure and then iterating the extended structure. For these arguments, therefore, we need to understand when a forcing extension of an iterable countable structure remains iterable. For a certain general class of forcing notions and embeddings, we shall show in Theorem 19 that indeed the lift of an iterable embedding to a forcing extension remains iterable, and what is more, lifting just the first step of the iteration to the forcing extension can lead to a lift of the entire iteration. In rather general circumstances, therefore, the iteration of a lift is a lift of the iteration.

This argument will rely on the following characterization of when an ultrapower of a forcing extension is a lift of the ultrapower of the ground model. Suppose that  $M$  is a transitive model of  $\text{ZFC}^-$ , that  $\mathbb{P}$  is a poset in  $M$  and that  $G \subseteq \mathbb{P}$  is  $M$ -generic. Suppose further that  $U$  is an  $M$ -ultrafilter on a cardinal  $\delta$  in  $M$  and  $U^*$  is an  $M[G]$ -ultrafilter extending  $U$ , both with well-founded ultrapowers. Then the ultrapower by  $U^*$  lifts the ultrapower by  $U$  if and only if every  $f : \delta \rightarrow M$  in  $M[G]$  is  $U^*$ -equivalent to some  $g : \delta \rightarrow M$  in  $M$ . For the forward direction, suppose that the ultrapower  $j : M[G] \rightarrow N^*$  by  $U^*$  lifts the ultrapower  $j : M \rightarrow N$  by  $U$  and  $\tau_G = f : \delta \rightarrow M$  is a function in  $M[G]$ . Note that  $f : \delta \rightarrow A$  where  $A = \{a \in M \mid \exists p \in \mathbb{P} \exists \xi \in \delta \, p \Vdash \tau(\xi) = \check{a}\}$  is an element of  $M$  by replacement. Thus,  $j(f)(\delta) \in j(A) \subseteq N$  and so  $j(f)(\delta) = j(g)(\delta)$  for some  $g \in M$ , from which it follows that  $f$  is  $U^*$ -equivalent to  $g$ . For the backward direction, note that there is an isomorphism between  $N$  and a transitive submodel of  $N^*$  sending  $[f]_U$  to  $[f]_{U^*}$ . Applying this characterization, if we lift the first embedding in the iteration, then the ultrafilter derived from the lift will have the above property. The key to the argument will be to capture this property as a schema of first-order statements over the forcing extension and propagate it along the iteration using elementarity.

Let us now discuss a class of posets for which this strategy proves successful. Suppose  $j : M \rightarrow N$  is an elementary embedding with critical point  $\delta$ . We define that a poset  $\mathbb{P} \in M$  is *j-useful* if  $\mathbb{P}$  is  $\delta$ -c.c. in  $M$  and  $j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{P}}_{\text{tail}}$ , where  $\mathbb{1}_{\mathbb{P}} \Vdash \text{"}\dot{\mathbb{P}}_{\text{tail}} \text{ is } \leq \delta\text{-strategically closed"}$  in  $N$ . There are numerous examples of such posets arising in the context of forcing with large cardinals, and we shall mention several in Sections 4 and 5. We presently explain how the property of *j-usefulness* allows us to find lifts of an ultrapower embedding to the forcing extension, so that the iteration of the lift is the lift of the iteration. If  $\mathbb{Q}$  is any poset and  $X$  is a set, not necessarily transitive, define as usual that a condition  $q \in \mathbb{Q}$  is *X-generic for*  $\mathbb{Q}$  if for every  $V$ -generic filter  $G \subseteq \mathbb{Q}$  containing  $q$  and every maximal antichain  $A \subseteq \mathbb{Q}$  with  $A \in X$ , the intersection  $G \cap A \cap X \neq \emptyset$ ; in other words,  $q$  forces over

$V$  that the generic filter meets the maximal antichains of  $X$  inside  $X$ . Suppose  $j : M \rightarrow N$  and  $\mathbb{P}$  is  $j$ -useful. Our key observation about  $j$ -usefulness is that if  $X \in N$  is sufficiently elementary in  $N$  with  $X^{<\delta} \subseteq X$  and  $|X| = \delta$  in  $N$ , then every condition  $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{P}}_{\text{tail}} \cap X$  can be strengthened to an  $X$ -generic condition. First, observe that every condition in  $\mathbb{P}$  is  $X$ -generic for  $\mathbb{P}$ , since maximal antichains of  $\mathbb{P}$  have size less than  $\delta$  and so if  $X$  contains such an antichain as an element, it must be a subset as well. Thus, for the pair  $(p, \dot{q})$  to be  $X$ -generic for  $j(\mathbb{P})$ , it suffices for  $p$  to force that  $\dot{q}$  sits below some element of every dense subset of  $\dot{\mathbb{P}}_{\text{tail}}$  in  $X[\dot{G}]$ . Such a  $\dot{q}$  is found by a simple diagonalization argument, using the facts that  $|X| = \delta$ ,  $X^{<\delta} \subseteq X$  and  $\dot{\mathbb{P}}_{\text{tail}}$  is forced to be  $\leq \delta$ -strategically closed.

Let us use the notation  $\langle M, \delta, U \rangle \models \text{“I am } H_{\delta+} \text{”}$  to mean that  $M$  believes every set has size at most  $\delta$ . We now prove that if a certain external genericity condition is met, then the iteration of a lift is a lift of the iteration.

**Theorem 19.** *Suppose that  $\langle M_0, \delta, U_0 \rangle \models \text{“I am } H_{\delta+} \text{”}$  is iterable, and that the first step of the iteration  $j_{01} : M_0 \rightarrow M_1$  lifts to an embedding  $j_{01}^* : M_0[G_0] \rightarrow M_1[G_1]$  on the forcing extension, where  $G_0 \subseteq \mathbb{P}$  is  $M_0$ -generic,  $j_{01}^*(G_0) = G_1 \subseteq j_{01}(\mathbb{P})$  is  $M_1$ -generic and  $\mathbb{P}$  is  $j_{01}$ -useful.*

$$\begin{array}{ccccccc} M_0[G_0] & \xrightarrow{j_{01}^*} & M_1[G_1] & & & & \\ \cup & & \cup & & & & \\ M_0 & \xrightarrow{j_{01}} & M_1 & \xrightarrow{j_{12}} & \dots & \xrightarrow{j_{\xi\xi+1}} & \dots \end{array}$$

Then  $j_{01}^*$  is the ultrapower by a weakly amenable  $M_0[G_0]$ -ultrafilter  $U_0^*$  extending  $U_0$ . Furthermore, if  $G_1$  meets certain external dense sets  $D_a \subseteq j_{01}(\mathbb{P})$  for  $a \in M_0$  described in the proof below, then  $\langle M_0[G_0], \delta, U_0^* \rangle$  is iterable, and the entire iteration of  $\langle M_0[G_0], \delta, U_0^* \rangle$  lifts the iteration of  $\langle M_0, \delta, U_0 \rangle$  step-by-step.

$$\begin{array}{ccccccc} M_0[G_0] & \xrightarrow{j_{01}^*} & M_1[G_1] & \xrightarrow{j_{12}^*} & \dots & \xrightarrow{j_{\xi\xi+1}^*} & \dots \\ \cup & & \cup & & & \cup & \\ M_0 & \xrightarrow{j_{01}} & M_1 & \xrightarrow{j_{12}} & \dots & \xrightarrow{j_{\xi\xi+1}} & \dots \end{array}$$

Thus, the iteration of the lift is a lift of the iteration.

*Proof.* Suppose that the ultrapower  $j_{01} : M_0 \rightarrow M_1$  by  $U_0$  lifts to  $j_{01}^* : M_0[G_0] \rightarrow M_1[G_1]$ , with  $j_{01}^*(G_0) = G_1$ . By the normality of  $U_0$ , it follows that every element of  $M_1$  has the form  $j_{01}(f)(\delta)$  for some  $f \in M_0^\delta \cap M_0$ . Every element of  $M_1[G_1]$  is  $\tau_{G_1}$  for some  $j_{01}(\mathbb{P})$ -name  $\tau \in M_1$ , and so  $\tau = j_{01}(t)(\delta)$  for some function  $t \in M_0$ . Define a function  $f$  in  $M_0[G_0]$  by  $f(\alpha) = t(\alpha)_{G_0}$ , and observe that  $j_{01}^*(f)(\delta) = j_{01}(t)(\delta)_{j_{01}^*(G_0)} = \tau_{G_1}$ . Thus, every element of  $M_1[G_1]$  has the form  $j_{01}^*(f)(\delta)$  for some  $f \in M_0[G_0]^\delta \cap M_0[G_0]$ . It follows that  $j_{01}^*$  is the ultrapower of  $M_0[G_0]$  by the  $M[G_0]$ -ultrafilter  $U_0^* = \{ X \subseteq \delta \mid X \in M_0[G], \delta \in j_{01}^*(X) \}$ , which extends  $U_0$ . Note that since  $\mathbb{P}$  is  $j_{01}$ -useful, it follows that  $j_{01}(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{P}}_{\text{tail}}$ , where  $\dot{\mathbb{P}}_{\text{tail}}$  adds no new subsets of  $\delta$  and  $\mathbb{P}$  is  $\delta$ -c.c. From this, we obtain that  $P(\delta)^{M_1[G_1]} = P(\delta)^{M_1[G_0]} = P(\delta)^{M_0[G_0]}$ , and so  $U_0^*$  is weakly amenable to  $M_0[G_0]$ . It therefore makes sense to speak of the iterated ultrapowers of  $\langle M_0[G_0], \delta, U_0^* \rangle$ , apart from the question of whether these iterates are well-founded.

The fact that the ultrapower  $j_{01} : M_0 \rightarrow M_1$  by  $U_0$  lifts to the ultrapower  $j_{01}^* : M_0[G_0] \rightarrow M_1[j_{01}^*(G_0)]$  by  $U_0^*$  is exactly equivalent to the assertion that for every function  $f \in M_0^\delta \cap M_0[G_0]$  there is a function  $g \in M_0^\delta \cap M_0$  such that  $f$  and  $g$  agree

on a set in  $U_0^*$ . In slogan form: Every new function agrees with an old function. This property is first order expressible in the expanded structure  $\langle M_0[G_0], \delta, U_0^*, M \rangle$ , by a statement with complexity at most  $\Pi_2$ . If  $j_{01}^*$  were sufficiently elementary on this structure, then it would preserve the truth of this statement and we could deduce easily that the iterates of  $U_0^*$  are step-by-step lifts of the corresponding iterates of  $U_0$ , completing the proof. Unfortunately, in the general case we cannot be sure that  $j_{01}^*$  is sufficiently elementary on this expanded structure. Similarly, although the original embedding  $j_{01} : M_0 \rightarrow M_1$  is fully elementary, it may not be fully elementary on the corresponding expanded structure  $j_{01} : \langle M_0, \delta, U_0 \rangle \rightarrow \langle M_1, \delta_1, U_1 \rangle$ . The rest of this argument, therefore, will be about getting around this difficulty by showing that if  $G_1$  satisfies an extra genericity criterion, then the iteration of  $U_0^*$  does indeed lift the iteration of  $U_0$ .

Specifically, through this extra requirement on  $G_1$ , we will arrange that for every  $a \in M_0$ , there is a set  $m_a \in M_0$  such that

- (1)  $m_a$  is a transitive model of  $\text{ZFC}^-$  containing  $\mathbb{P}$  and  $a$ , and
- (2) every  $f : \delta \rightarrow m_a$  in  $m_a[G_0]$  is  $u_a^*$ -equivalent to some  $g : \delta \rightarrow m_a$  in  $m_a$ ,

where  $u_a^* = m_a \cap U_0^*$ , which is an element of  $M_0[G_0]$  by the weak amenability of  $U_0^*$  to  $M_0[G_0]$ .

Let us first suppose that we have already attained (1) and (2) for every  $a$  and explain next how this leads to the conclusion of the theorem. Suppose inductively that the iteration of  $U_0^*$  on  $M_0[G_0]$  is a step-by-step lift of the iteration of  $U_0$  on  $M_0$  up to stage  $\xi$ . Note that limit stages come for free, because if every successor stage before a limit is a lift, then the limit stage is also a lift. Thus, we assume that the diagram in the statement of the theorem is accurate through stage  $\xi$ , so that in particular the  $\xi^{\text{th}}$  iteration  $j_{0\xi}^* : M_0[G_0] \rightarrow M_\xi[G_\xi]$  of  $U_0^*$  is a lift of the  $\xi^{\text{th}}$  iteration  $j_{0\xi} : M_0 \rightarrow M_\xi$  of  $U_0$ , and we consider the next step  $M_\xi[G_\xi] \rightarrow \text{Ult}(M_\xi[G_\xi], U_\xi^*)$ . Since any given instance of (1) and (2), for fixed  $a$ , is expressible in  $M_0[G_0]$  as a statement about  $(m_a, G_0, u_a^*, a, \mathbb{P})$ , it follows by elementarity that  $j_{0\xi}^*(m_a)$  is a transitive model of  $\text{ZFC}^-$  containing  $j_{0\xi}(\mathbb{P})$ , and that every  $f : j_{0\xi}^*(\delta) \rightarrow j_{0\xi}^*(m_a)$  in  $j_{0\xi}^*(m_a)[G_\xi]$  is  $j_{0\xi}^*(u_a^*)$ -equivalent to a function  $g : j_{0\xi}(\delta) \rightarrow j_{0\xi}(m_a)$  in  $j_{0\xi}(m_a)$ . Note that since  $u_a^* \subseteq U_0^*$ , it follows by an easy argument that  $j_{0\xi}^*(u_a^*) \subseteq U_\xi^*$ . Thus, as far as  $j_{0\xi}^*(m_a)$  and  $j_{0\xi}^*(m_a)[G_\xi]$  are concerned, every new function agrees with an old function. But now the key point is that the  $j_{0\xi}(m_a)$  exhaust  $M_\xi$ , since every object in  $M_\xi$  has the form  $j_{0\xi}(f)(s)$  for some finite  $s \subseteq \delta_\xi$ , and thus once we put  $f$  into  $m_a$  by a suitable choice of  $a$ , then  $j_{0\xi}(f)(s)$  will be in  $j_{0\xi}(m_a)$ . From this, it follows that the  $j_{0\xi}^*(m_a[G_0])$  exhaust  $M_\xi[G_\xi]$ , since every element of  $M_\xi[G_\xi]$  has a name in  $M_\xi$ . Therefore, every new function in  $M_\xi[G_\xi]$  agrees on a set in  $U_\xi^*$  with an old function in  $M_\xi$ , and so the ultrapower of  $M_\xi[G_\xi]$  by  $U_\xi^*$  is a lift of  $j_{\xi\xi+1}$ . Thus, we have continued the step-by-step lifting one additional step, and so by induction, the entire iteration lifts step-by-step as claimed.

It remains to explain how we achieve (1) and (2) for every  $a \in M_0$ . First, we observe that  $M_0$  is the union of transitive models  $m$  of ZFC. This is because any set  $A \subseteq \delta$  in  $M_0$  is also in  $M_1$  and therefore in  $V_{j_{01}(\delta)}^{M_1}$ , which is a model of ZFC since  $j_{01}(\delta)$  is inaccessible in  $M_1$ . By collapsing an elementary substructure of this structure in  $M_1$ , therefore, we find a size  $\delta$  transitive model  $m \models \text{ZFC}$  with  $A \in m \in M_1$ . Since  $m$  has size  $\delta$  and  $M_0 = H_{\delta^+}^{M_1}$  by weak amenability, it follows

that  $m \in M_0$  as well. Thus, for any  $a \in M_0$  there are numerous models  $m$  as in Statement (1), even with full ZFC.

For any such  $m$ , let  $X_m = \{j_{01}(f)(\delta) \mid f \in m\}$ . It is not difficult to check that  $X_m \prec j_{01}(m)$ , by verifying the Tarski-Vaught criterion. Also, since  $j_{01} \upharpoonright m \in M_1$ , it follows that  $X_m \in M_1$ , although by replacement the map  $m \mapsto X_m$  cannot exist in  $M_1$ , since  $M_1$  is the union of all  $X_m$ . For any  $a \in M_0$ , let

$$D_a = \{q \in j_{01}(\mathbb{P}) \mid q \text{ is } X_m\text{-generic for some transitive } m \models \text{ZFC}^- \text{ with } a \in m \in M_0\}.$$

Recall that a condition  $q$  is  $X_m$ -generic for  $j_{01}(\mathbb{P})$  if every  $M_1$ -generic filter  $G \subseteq j_{01}(\mathbb{P})$  has  $G \cap D \cap X_m \neq \emptyset$  for every dense set  $D \subseteq j_{01}(\mathbb{P})$  in  $M_1$ . Because the definition of  $D_a$  refers to the various  $X_m$ , there is little reason to expect that  $D_a$  is a set in  $M_1$ . Nevertheless, we shall argue anyway that it is a dense subset of  $j_{01}(\mathbb{P})$ .

To see this, fix  $a$  and any condition  $p \in j_{01}(\mathbb{P})$ . Since  $p = j(\vec{p})(\delta)$  for some function  $\vec{p} \in M_0$ , we may find as we explained above a transitive set  $m \in M_0$  with  $a, \vec{p}, \mathbb{P} \in m \models \text{ZFC}$ . We may also ensure in that argument that  $m^{<\delta} \subseteq m$  in  $M_0$ . It follows that  $X_m^{<\delta} \subseteq X_m$  in  $M_1$ , and since  $\vec{p} \in m$ , we also know that  $p = j(\vec{p})(\delta) \in X_m$ . The forcing  $j_{01}(\mathbb{P})$  is in  $X_m$  and factors as  $\mathbb{P} * \dot{\mathbb{P}}_{\text{tail}}$ , where  $\mathbb{P}$  is  $\delta$ -c.c. and  $\dot{\mathbb{P}}_{\text{tail}}$  is forced to be  $\leq \delta$ -strategically closed. Since  $M_1$  knows that  $X_m$  has size  $\delta$ , it can perform a diagonalization below  $p$  of the dense sets for the tail forcing, and thereby produce a  $\mathbb{P}$ -name for a condition in  $\dot{\mathbb{P}}_{\text{tail}}$  meeting all those dense sets. (This is where we have used the key property of  $j$ -usefulness mentioned before the theorem.) Thus,  $M_1$  can build an  $X_m$ -generic condition  $q$  for  $j_{01}(\mathbb{P})$  below  $p$ . This establishes that  $D_a$  is dense, as we claimed.

We now suppose that  $G_1$  meets all the dense sets  $D_a$ , and use this to establish (1) and (2). For any  $a \in M_0$ , we have a condition  $q \in G_1$  that is  $X_m$ -generic for some transitive  $m \models \text{ZFC}^-$  in  $M_0$  containing  $\langle a, \mathbb{P} \rangle$ , thereby satisfying (1). From this, it follows that  $X_m[G_1] \cap M_1 = X_m$ , since for any name in  $X_m$  for an object in  $M_1$ ,  $X_m$  has a dense set of conditions deciding its value, and since  $G_1$  meets this dense set inside  $X_m$ , the decided value must also be in  $X_m$ . Now, suppose that  $f : \delta \rightarrow m$  is a function in  $m[G_0]$ , so that  $f = \dot{f}_{G_0}$  for some name  $\dot{f} \in m$ . Since  $\dot{f} \in m$ , it follows that  $j_{01}(\dot{f}) \in X_m$ , and so  $j_{01}(f)(\delta) \in X_m[G_1]$ . Since  $\text{ran}(f) \subseteq m$ , it follows that  $\text{ran}(j_{01}(f)) \subseteq j_{01}(m)$ , which is contained in  $M_1$ . Thus,  $j_{01}(f)(\delta) \in X_m[G_1] \cap M_1$ , which is equal to  $X_m$ . But every element of  $X_m$  has the form  $j_{01}(g)(\delta)$  for some function  $g \in m$ , and so  $j_{01}(f)(\delta) = j_{01}(g)(\delta)$  for such a function  $g$ . It follows that  $f$  and  $g$  agree on a set in  $U_0^*$  and we have established (2), completing the argument.  $\square$

A special case of the theorem occurs when  $\mathbb{P}$  has size smaller than  $\delta$  in  $M_0$ . In this case,  $\dot{\mathbb{P}}_{\text{tail}}$  is trivial and the extra genericity condition is automatically satisfied, since the dense sets  $D_a$  would be elements of  $M_1$ . The nontrivial case of the theorem occurs when the forcing  $\mathbb{P}$  has size  $\delta$ , and its image is therefore stretched on the ultrapower side. We are unsure about the extent to which it could be true generally that the iteration of a lift is a lift of the iteration. Surely some hypotheses are needed on the forcing, since if  $\mathbb{P}$  is an iteration of length  $\delta$  and  $j(\mathbb{P})$  adds new subsets to  $\delta$  at stage  $\delta$ , for example, then the lift  $j_{01}^*$  will not be weakly amenable, making it impossible to iterate. Our  $j$ -usefulness hypothesis avoids this issue, but we are not sure whether it is possible to omit the external genericity assumption we made on  $G_1$ . Nevertheless, this extra genericity assumption appears to be no



more difficult to attain in practice than ordinary  $M_1$ -genericity. For example, in the case of countable structures:

**Corollary 20.** *If  $\langle M, \delta, U \rangle \models "I \text{ am } H_{\delta^+}"$  is a countable iterable structure and  $\mathbb{P} \in M$  is useful for the ultrapower of  $M$  by  $U$ , then there is an  $M$ -generic filter  $G \subseteq \mathbb{P}$  and  $M[G]$ -ultrafilter  $U^*$  extending  $U$  such that  $\langle M[G], \delta, U^* \rangle$  is iterable, and the iteration of  $M[G]$  by  $U^*$  is a step-by-step lift of the iteration of  $M$  by  $U$ .*

*Proof.* This is simply a special case of the previous theorem. When  $M$  is countable, then there is no trouble in finding an  $M$ -generic filter  $G$  and  $M_1$ -generic filter  $G_1$  satisfying the extra genericity requirement, since there are altogether only countably many dense sets to meet.  $\square$

#### 4. THE THIRD PROOF METHOD

In this section, for the third proof method, we generalize the proof sketch of Theorem 16 given at the end of Section 2. For the arguments here, we shall use the hypothesis of having a weakly iterable cardinal  $\delta$  with  $V_\delta$  a model containing large cardinals. We shall use the structure  $\langle M, \delta, U \rangle$  witnessing the weak iterability of  $\delta$  to produce a countable iterable structure and build the inner model out of the iterates of this structure or the iterates of its forcing extension.

**Theorem 21.** *If  $\langle M, \delta, U \rangle$  is iterable with a poset  $\mathbb{P} \in V_\delta^M$ , then there is an inner model satisfying every sentence forced by  $\mathbb{P}$  over  $V_\delta^M$ .*

*Proof.* Let  $\langle M_0, \delta_0, U_0 \rangle$  be obtained by collapsing a countable elementary substructure of  $\langle M, \delta, U \rangle$  containing  $\mathbb{P}$ . By Lemma 18,  $\langle M_0, \delta_0, U_0 \rangle$  is iterable. Also, if  $\mathbb{Q}$  is the collapse of the poset  $\mathbb{P}$ , then by elementarity  $\mathbb{Q}$  forces the same sentences over  $V_{\delta_0}^{M_0}$  that  $\mathbb{P}$  forces over  $V_\delta^M$ . Let  $\{j_{\xi\eta} : M_\xi \rightarrow M_\eta \mid \xi < \eta \in \text{Ord}\}$  be the corresponding directed system of iterated ultrapowers of  $M_0$ , and consider the inner model  $W = \bigcup_{\xi \in \text{Ord}} j_{0\xi}(V_{\delta_0}^{M_0})$ , which is the cumulative part of the iteration lying below the critical sequence. Since  $V_{\delta_0}^{M_0} \prec W$  and  $V_{\delta_0}^W = V_{\delta_0}^{M_0}$ , it follows that  $\mathbb{Q}$  forces the same sentences over  $V_{\delta_0}^{M_0}$  as over  $W$ , and these are the same as forced by  $\mathbb{P}$  over  $V_\delta^M$ . Since  $\mathbb{Q}$  lies below the critical point  $\delta_0$  of the iteration, the model  $W$  contains only countably many dense subsets of  $\mathbb{Q}$  and so we can build a  $W$ -generic filter  $G$  directly. Thus, the model  $W[G]$ , an inner model of  $V$ , satisfies the requirement of the theorem.  $\square$

Let us now apply this theorem to the case of an indestructible supercompact cardinal.

**Theorem 22.** *If  $\kappa$  is  $<\delta$ -supercompact for a weakly iterable cardinal  $\delta$  above  $\kappa$ , then there is an inner model in which  $\kappa$  is an indestructible supercompact cardinal.*

*Proof.* Suppose that  $\kappa$  is  $<\delta$ -supercompact for a weakly iterable cardinal  $\delta$  above  $\kappa$  and the weak iterability of  $\delta$  is witnessed by an iterable structure  $\langle M, \delta, U \rangle$ , with  $V_\delta \in M$ . In particular,  $\kappa$  is  $<\delta$ -supercompact in  $M$ . Note that the Laver preparation  $\mathbb{P}$  of  $\kappa$  is small relative to  $\delta$  in  $M$ . Thus, by Theorem 21, there is an inner model  $W_0$  satisfying the theory forced by  $\mathbb{P}$  over  $V_\delta$ . The forcing  $\mathbb{P}$ , of course, makes  $\kappa$  indestructibly supercompact in  $V_\delta^\mathbb{P}$ , and so the inner model  $W_0$  has an indestructible supercompact cardinal  $\kappa_0$ .

In order to prove the full claim, we must find a  $W$  in which  $\kappa$  itself is indestructibly supercompact. For this, let us look more closely at how the inner model

$W_0$  arises from the proof of Theorem 21. Specifically, the indestructible supercompact cardinal  $\kappa_0$  of  $W_0$  arises inside a countable iterable structure  $M_0$ , obtained by a Mostowski collapse of a countable structure containing  $\kappa$ , and  $\kappa_0$  is below the critical point  $\delta_0$  of the iteration. Thus,  $\kappa_0$  is not moved by the iteration and is therefore a countable ordinal in  $V$ , even though it is indestructibly supercompact in  $W_0$ . Since in particular  $\kappa_0$  is measurable in  $W_0$ , we may consider the internal system of embeddings obtained by iterating a normal measure on  $\kappa_0$  in  $W_0$ . The successive images of  $\kappa_0$  lead to the critical sequence  $\{\kappa_\alpha \mid \alpha \in \text{Ord}\}$ , which is a closed unbounded class of ordinals, containing all cardinals of  $V$ . It follows that  $\kappa$  itself appears on this critical sequence, as the  $\kappa^{\text{th}}$  element  $\kappa = \kappa_\kappa$ . In particular, if  $j : W_0 \rightarrow W_\kappa$  is the  $\kappa^{\text{th}}$  iteration of the normal measure, then  $j(\kappa_0) = \kappa$ , and so by elementarity,  $W_\kappa$  is an inner model in which  $\kappa$  itself is an indestructible supercompact cardinal.  $\square$

It should be clear that once there is an inner model  $W$  containing an indestructible supercompact cardinal, and this cardinal is a mere countable ordinal in  $V$ , then in fact it can be arranged that any desired cardinal of  $V$  is an indestructible supercompact cardinal in an inner model. For example, this argument shows that if there is a cardinal that is supercompact up to a weakly iterable cardinal, then there are inner models  $W$  in which  $\aleph_1^V$  is indestructibly supercompact, or  $\aleph_2^V$  or  $\aleph_\omega^V$  is indestructibly supercompact, and so on, as desired.

The method also provides an answer to Test Question 5.

**Theorem 23.** *If  $\kappa$  is  $<\delta$ -supercompact for a weakly iterable cardinal  $\delta$  above  $\kappa$ , then there is an inner model of the PFA.*

*Proof.* Let  $\langle M, \delta, U \rangle$  be an iterable structure containing  $V_\delta$ . Then  $\kappa$  is supercompact in  $V_\delta$ , and so the Baumgartner forcing  $\mathbb{P} \in V_\delta$  forces the PFA over  $V_\delta$ . Thus, by Theorem 21, there is an inner model of the PFA.  $\square$

Let us return to Test Question 1, where we aim to produce an inner model with a supercompact cardinal and the full GCH. In Theorem 14, we approached this, by finding inner models with a supercompact cardinal  $\kappa$  such that  $2^\kappa = \kappa^+$  or such that  $2^\kappa = \kappa^{++}$ , and the proof generalized to get various GCH patterns at or above  $\kappa$ . The proofs of those theorems, however, relied on the friendliness of the iteration up to  $\kappa$ , and so seem unable to attain the full GCH. For example, if CH fails in  $V$ , then there can be no friendly forcing of the GCH. The third proof method, however, does work to produce such an inner model. We cannot apply Theorem 21 directly to the case of the poset forcing the GCH, since it is a class forcing over  $V_\delta$ . Following the proof of Theorem 21, we would need at the last step to obtain a generic for a class forcing over the inner model  $W$ , and there is no obvious reason to suppose that such a  $W$ -generic can be constructed. Instead, using Theorem 19, we shall follow the modified strategy of forcing over the countable iterable structure first and then iterating out to produce the inner model. Note that if the GCH fails in  $V$ , then for large  $\theta$  one cannot expect to find the GCH in the robust type of inner models  $W$  for which  $W^\theta \subseteq W$ , since such a property would inject the GCH violations from  $V$  into  $W$ .

The following theorem generalizes Theorem 21 to the case of class forcing with respect to  $V_\delta$ .

**Theorem 24.** *If  $\langle M, \delta, U \rangle$  is iterable and  $\mathbb{P} \subseteq V_\delta^M$  is a poset in  $M$  and useful for the ultrapower by  $U$ , then there is an inner model satisfying every sentence forced by  $\mathbb{P}$  over  $V_\delta^M$ .*

*Proof.* We may assume without loss of generality that  $\langle M, \delta, U \rangle \models \text{“I am } H_{\delta^+}\text{”}$ . (If not, replace  $M$  with  $H_{\delta^+}^M$  and observe that the structure  $\langle H_{\delta^+}^M, \delta, U \rangle$  remains iterable since it has all the same functions  $f : \delta \rightarrow H_{\delta^+}^M$  as  $M$  and therefore its iterates are substructures of the corresponding iterates of  $\langle M, \delta, U \rangle$ .) As in Theorem 21, let  $\langle M_0, \delta_0, U_0 \rangle$  be a countable iterable structure obtained by collapsing a countable elementary substructure of  $\langle M, \delta, U \rangle$  containing  $\mathbb{P}$ , and let  $\mathbb{Q}$  be the image of  $\mathbb{P}$  under the collapse. Since  $M_0$  is countable, there is by Corollary 20 an  $M_0$ -generic filter  $G_0 \subseteq \mathbb{Q}$  and an  $M_0[G_0]$ -ultrafilter  $U_0^*$  extending  $U_0$  such that  $\langle M_0[G_0], \delta_0, U_0^* \rangle$  is iterable, and such that the iteration of  $M_0[G_0]$  by  $U_0^*$  is a step-by-step lifting of the iteration of  $M_0$  by  $U_0$ . Note that  $V_{\delta_0}^{M_0[G_0]} = V_{\delta_0}^{M_0}[G_0]$  satisfies the theory forced by  $\mathbb{P}$  over  $V_\delta^M$ . Let  $\{j_{\xi\eta} : M_\xi[G_\xi] \rightarrow M_\eta[G_\eta] \mid \xi < \eta \in \text{Ord}\}$  be the directed system of iterated ultrapowers of  $M_0[G_0]$ , and consider  $W = \bigcup_{\xi \in \text{Ord}} j_{0\xi}(V_{\delta_0}^{M_0[G_0]})$ . Since the iteration of  $U_0^*$  lifts the iteration of  $U_0$  on  $M_0$  step-by-step, it follows that  $W = \bar{W}[H]$ , where  $\bar{W} = \bigcup_{\xi \in \text{Ord}} j_{0\xi}(V_{\delta_0}^{M_0})$  and  $H$  is the  $\bar{W}$ -generic filter arising from  $\bigcup_{\xi} j_{0\xi}(G_0)$  for the class forcing obtained by  $\bigcup_{\xi} j_{0\xi}(\mathbb{Q})$ . By elementarity,  $W$  satisfies the same sentences that are forced to hold over  $V_{\delta_0}^{M_0}$  by  $\mathbb{Q}$ , and these are the same as those forced to hold over  $V_\delta^M$  by  $\mathbb{P}$ .  $\square$

We may now apply Theorem 24 to provide answers to Test Questions 1 and 6, from a stronger hypothesis.

**Theorem 25.** *If  $\kappa$  is  $<\delta$ -supercompact for a weakly iterable cardinal  $\delta$ , then there is an inner model in which  $\kappa$  is supercompact and the GCH plus  $V = \text{HOD}$  hold.*

*Proof.* Let  $\langle M, \delta, U \rangle$  be an iterable structure containing  $V_\delta$ , and as before, assume without loss of generality that  $M \models \text{“I am } H_{\delta^+}\text{”}$ . Observe that the canonical class forcing of the GCH is definable over  $V_\delta$  and useful for the ultrapower embedding. Note that although  $\delta$  may be singular in  $V$ , it is Mahlo (and more) in  $M$ , and so the forcing is  $\delta$ -c.c. inside  $M$ . By Theorem 24, there is an inner model with a supercompact cardinal and the GCH. To obtain an inner model where  $\kappa$  itself is supercompact, simply follow the second part of the proof of Theorem 22. One can similarly obtain an inner model satisfying  $V = \text{HOD}$  without the GCH by coding sets into the continuum function, making essentially the same argument. (See, e.g., the coding method used in [Rei06, Theorem 11] or [Rei07, Theorem 11].) If  $\text{GCH} + V = \text{HOD}$  is desired, as in the statement of the theorem, then one should use a coding method compatible with the GCH. For example, the  $\diamond_\gamma^*$  coding method used in [BT09], in conjunction with the proof of [Rei06, Theorem 11] or [Rei07, Theorem 11], forces  $\text{GCH} + V = \text{HOD}$  while preserving supercompactness, and has the desired closure properties for this argument.  $\square$

The hypotheses of Theorems 22, 23 and 25 can be improved slightly, since it is not required that  $\delta$  is weakly iterable, but rather only that

(\*)  $\kappa$  is  $<\delta$ -supercompact inside an iterable structure  $\langle M, \delta, U \rangle$  where  $V_\delta^M$  exists.

It is irrelevant assuming (\*) whether  $V_\delta^M$  is the true  $V_\delta$ , since the only use of that in our argument was to ensure that  $\kappa$  was  $<\delta$ -supercompact in  $M$ .

Next, we improve the iteration method to find more robust inner models, which not only satisfy the desired theory, but which also agree with  $V$  up to  $\delta$ . This sort of additional feature cannot be attained by iterating a countable model out of the universe, which is ultimately how our earlier instances of the iteration method proceeded.

Suppose as usual that  $\langle M, \delta, U \rangle$  is a structure where  $M \models \text{ZFC}^-$ ,  $\delta$  is a cardinal in  $M$ , and  $U$  is a weakly amenable  $M$ -ultrafilter. Suppose further that  $V_\delta^M$  exists. As a shorthand, let us refer to these structures as *weakly amenable*. A weakly amenable structure that is closed under  $<\delta$ -sequences is automatically iterable. This is because it will be correct about the countable completeness of the ultrafilter, which suffices for iterability (see [Kun70]). Moreover, closure under  $<\delta$ -sequences implies that  $\delta$  is inaccessible and hence  $V_\delta^M = V_\delta$ . Thus, if there exists a weakly amenable structure with  $M^{<\delta} \subseteq M$ , then  $\delta$  is weakly iterable. The existence of these structures, however, has a significantly larger consistency strength than the existence of a weakly iterable cardinal that is between Ramsey and measurable cardinals (see [Git11]).

**Theorem 26.** *Suppose  $\langle M, \delta, U \rangle$  is weakly amenable with  $M^{<\delta} \subseteq M$ . Suppose that  $\mathbb{P} \subseteq V_\delta$  is a poset in  $M$  such that for every  $\gamma < \delta$ , there is a condition  $p \in \mathbb{P}$  such that  $\mathbb{P} \restriction p$  is  $\leq \gamma$ -strategically closed and useful for the ultrapower of  $M$  by  $U$ . Then there is an inner model  $W$  of  $V$  satisfying every sentence forced by  $\mathbb{P}$  over  $V_\delta$  and with  $V_\delta^W = V_\delta$ .*

*Proof.* As usual, without loss of generality, we assume that  $M \models \text{"I am } H_{\delta^+}\text{"}$ . We may also assume that  $M$  has size  $\delta$ , since if necessary, we may replace  $M$  with an elementary substructure  $M^* = \bigcup_\omega M_n$ , where each  $M_n \prec M$  of size  $\delta$  is constructed in the ultrapower so that  $M_n \cap U \in M_{n+1}$ , and observe that the structure  $\langle M^*, \delta, U \rangle$  remains iterable by Lemma 18. The hypothesis on  $\mathbb{P}$  is a superfriendly version of usefulness. Consider the first two steps of the iteration

$$M = M_0 \xrightarrow{j_{01}} M_1 \xrightarrow{j_{12}} M_2.$$

Our strategy will be to lift the *second* step of the iteration. We shall produce in  $V$  a lift  $j_{12}^* : M_1[G_1] \rightarrow M_2[G_2]$ , where  $G_1 \subseteq j_{01}(\mathbb{P})$  is  $M_1$ -generic and  $j_{12}^*(G_1) = G_2$  is  $M_2$ -generic for  $j_{02}(\mathbb{P})$ , while also satisfying the extra genericity requirement of Theorem 19. By that theorem, therefore, the lift will be iterable and the desired inner model will be obtained by iterating it out of the universe.

To begin, note that the structure  $\langle M_1, \delta_1, U_1 \rangle$  arising from the ultrapower of  $\langle M, \delta, U \rangle$  is certainly iterable, since it was obtained after one step of the iterable structure  $\langle M, \delta, U \rangle$ . In addition, the assumptions on  $M_0$  ensure that  $M_1^{<\delta} \subseteq M_1$  and  $|M_1| = \delta$ , and also that  $M_2^{<\delta} \subseteq M_2$  and  $|M_2| = \delta$ . By the superfriendly assumption on  $\mathbb{P}$ , and using elementarity, we may find a condition  $p \in j_{01}(\mathbb{P})$  below which  $j_{01}(\mathbb{P})$  is  $<\delta$ -strategically closed in  $M_1$ , and hence truly  $<\delta$ -strategically closed. By definition of usefulness,  $\mathbb{P}$  has  $\delta$ -c.c. in  $M_0$  and  $j_{01}(\mathbb{P})$  factors in  $M_1$  as  $\mathbb{P} * \dot{\mathbb{P}}_{\text{tail}}$  with  $\dot{\mathbb{P}}_{\text{tail}}$  forced to be  $\leq \delta$ -strategically closed. By elementarity, it follows that  $j_{01}(\mathbb{P})$  has  $j_{01}(\delta)$ -c.c. in  $M_1$  and  $j_{01}(\mathbb{P})$  factors in  $M_2$  as  $j_{01}(\mathbb{P}) * j_{01}(\dot{\mathbb{P}}_{\text{tail}})$  with  $j_{01}(\dot{\mathbb{P}}_{\text{tail}})$  forced to be  $\leq j_{01}(\delta)$ -strategically closed, and hence  $j_{01}(\mathbb{P})$  is useful

for  $j_{12}$ . It follows that below the condition  $(p, \dot{\mathbb{I}}_{j_{01}(\dot{\mathbb{P}}_{\text{tail}})})$ , the poset  $j_{02}(\mathbb{P})$  is  $<\delta$ -strategically closed. Since there are only  $\delta$  many dense subsets of  $j_{02}(\mathbb{P})$  in  $M_2$  and  $M_2^{<\delta} \subseteq M_2$ , we may diagonalize to find an  $M_2$ -generic filter  $G_2 \subseteq j_{02}(\mathbb{P})$  below  $p$  in  $V$ . It follows that  $G_1 = G_2 \restriction j_{01}(\delta)$  is  $M_1$ -generic for  $j_{01}(\mathbb{P})$ , and we may lift the embedding  $j_{12}$  to  $j_{12}^* : M_1[G_1] \rightarrow M_2[G_2]$ . We may furthermore arrange in the diagonalization that  $G_2$  also meets all the external dense sets  $D_a$  arising in Theorem 19, since there are only  $\delta$  many such additional sets, and they can simply be folded into the diagonalization. Thus, by Theorem 19, the lift  $j_{12}^*$  is iterable. Let  $\{j_{1\xi}^* : M_1[G_1] \rightarrow M_\xi[G_\xi]\}$  be the corresponding iteration, and let  $W = \bigcup_\xi V_{j_{1\xi}(\delta_1)}^{M_\xi[G_\xi]}$  be the resulting inner model. This is the union of an elementary chain, and so  $W$  is an elementary extension of  $M_1[G_1]$ , which satisfies all sentences forced by  $\mathbb{P}$  over  $V_\delta$  and includes  $H_{\delta^+}^M$ . In particular,  $V_\delta \subseteq W$  and so  $V_\delta^W = V_\delta$ , completing the proof.  $\square$

**Theorem 27.** *If  $\kappa$  is indestructibly  $<\delta$ -supercompact in a weakly amenable  $\langle M, U, \delta \rangle$  with  $M^{<\delta} \subseteq M$ , then there is an inner model  $W$  satisfying  $V = \text{HOD}$  in which  $\kappa$  is indestructibly supercompact and for which  $V_\delta^W = V_\delta$ .*

*Proof.* Let  $\langle M, \delta, U \rangle$  be weakly amenable with  $M^{<\delta} \subseteq M$ . It follows that  $\kappa$  is indestructibly supercompact in  $V_\delta$ . Let  $\mathbb{P}$  be the forcing notion that first generically chooses (via a lottery sum) an ordinal  $\gamma_0$  in the interval  $[\kappa, \delta)$ , and then performs an Easton support iteration of length  $\delta$ .  $\mathbb{P}$  does nontrivial forcing at regular cardinals  $\gamma$  in the interval  $[\gamma_0, \delta)$ , with forcing that either forces the GCH to hold at  $\gamma$  or to fail at  $\gamma$ , using the lottery sum  $\oplus \{\text{Add}(\gamma^+, 1), \text{Add}(\gamma, \gamma^{++})\}$ . An easy density argument (see the proof of [Fri09, Lemma 13.1]) shows that any particular set of ordinals below  $\delta$  added by this forcing will be coded into the GCH pattern below  $\delta$ , and so  $\mathbb{P}$  forces  $V = \text{HOD}$  over  $V_\delta$ . By indestructibility, the forcing  $\mathbb{P}$  preserves the indestructible supercompactness of  $\kappa$ . Furthermore, the forcing  $\mathbb{P}$  is definable in  $V_\delta$ , and the choice of  $\gamma_0$  makes the forcing as closed as desired below  $\delta$ , as well as useful for the ultrapower of  $M$  by  $U$ . Thus, the hypotheses of Theorem 26 are satisfied. So by that theorem, there is an inner model  $W$  satisfying  $V = \text{HOD}$  and having  $V_\delta^W = V_\delta$ . Since  $\kappa$  is below  $\delta$ , the critical point of the iteration of  $M$  by  $U$ , it is not moved by that iteration, and so  $\kappa$  is indestructibly supercompact in  $W$ .  $\square$

Next, we consider a variant of one of the questions mentioned after Test Question 7, asking the extent to which sets can be placed into the HOD of an inner model.

**Theorem 28.** *If  $\kappa$  is strongly Ramsey, then for any  $A \in H_{\kappa^+}$ , there is an inner model  $W$  containing  $A$  and satisfying  $V = \text{HOD}$ . If the GCH holds below  $\kappa$ , then one can arrange that  $A$  is definable in  $W$  without parameters.*

*Proof.* From our earlier discussion, we know that  $\kappa$  is strongly Ramsey if every  $A \in H_{\kappa^+}$  can be placed into a weakly amenable structure  $\langle M, \kappa, U \rangle$  with  $M^{<\kappa} \subseteq M$ .

Starting with a weakly amenable  $\langle M, \kappa, U \rangle$  with  $M^{<\kappa} \subseteq M$  and  $A \in M$ , we use the same forcing as in the proof of Theorem 27 and appeal to Theorem 26 to obtain an inner model  $W$  satisfying  $V = \text{HOD}$  and having  $A \in W$ , as desired.

Lastly, if the GCH holds below  $\kappa$ , then as in Theorem 15, we may arrange the coding to begin with coding  $A$ , and thereby make  $A$  definable in  $W$  without parameters.  $\square$

**Corollary 29.** *If there is a proper class of strongly Ramsey cardinals, then every set  $A$  is an element of some inner model  $W$  satisfying  $V = \text{HOD}$ .*

*Proof.* Under this hypothesis, every set  $A$  is in  $H_{\delta^+}$  for some strongly Ramsey cardinal  $\delta$ , and so is in an inner model  $W$  satisfying  $V = \text{HOD}$  by Theorem 28.  $\square$

To summarize the situation with our test questions, we have provided definite affirmative answers to Test Questions 2, 3, 4 and 7, along with several variants, but have only provided the affirmative conclusion of Test Questions 1, 5 and 6 from the (consistency-wise) stronger hypothesis that there is a cardinal supercompact up to a weakly iterable cardinal (or at least supercompact inside an iterable structure). We do not know if this hypothesis can be weakened for these results to merely a supercompact cardinal. Perhaps either Woodin's new approach to building non-fine-structural inner models of a supercompact cardinal, or Foreman's approach of [For09] for constructing inner models of very large cardinals, will provide the answers to these questions.

## 5. FURTHER APPLICATIONS

We shall now describe how variants of our methods can be used to obtain a further variety of inner models. First, using the methods of Theorem 25 and a stronger hypothesis, we can obtain:

**Theorem 30.** *Suppose that  $\delta$  is a weakly iterable cardinal and a limit of cardinals that are  $<\delta$ -supercompact. Then:*

- (1) *There is an inner model with a proper class of supercompact cardinals, all of which are indestructible.*
- (2) *There is an inner model with a proper class of supercompact cardinals, where the GCH holds.*
- (3) *There is an inner model with a proper class of supercompact cardinals, where  $V = \text{HOD}$  and the Ground Axiom hold.*

Of course, there are numerous other possibilities, for any of the usual forcing iterations; we mention only these three as representative. In each case, the natural forcing has the same closure properties needed to support the argument of Theorem 25. In the case of Statement (2), for example, one uses the canonical Easton support forcing of the GCH, and in Statement (3), one uses any of the usual iterations that force every set to be coded into the GCH pattern of the continuum function, a state of affairs that implies both  $V = \text{HOD}$  and the Ground Axiom, the assertion that the universe was not obtained by set forcing over any inner model (see [Ham05], [Rei06] and [Rei07]).

For the next application of our methods, we show that there are inner models witnessing versions of classical results of Magidor [Mag76].

**Theorem 31.**

- (1) *If  $\kappa$  is  $<\delta$ -strongly compact for a weakly iterable cardinal  $\delta$  above  $\kappa$ , then there is an inner model in which  $\kappa$  is both the least strongly compact and the least measurable cardinal.*
- (2) *If there is a strongly compact cardinal  $\kappa$ , then there is an inner model in which the least strongly compact cardinal has only boundedly many measurable cardinals below it.*

- (3) *If there is a supercompact cardinal  $\kappa$ , then for every cardinal  $\theta$ , there is an inner model  $W$  in which the least strongly compact cardinal is the least supercompact cardinal and for which  $W^\theta \subseteq W$ .*
- (4) *If  $\kappa$  is  $<\delta$ -supercompact for a weakly iterable cardinal  $\delta$  above  $\kappa$ , then there is an inner model in which  $\kappa$  is both the least strongly compact and least supercompact cardinal.*

*Proof.* For (1), let  $\mathbb{P}$  be Magidor's notion of iterated Prikry forcing from [Mag76], which adds a Prikry sequence to every measurable cardinal below  $\kappa$ . Since  $|\mathbb{P}| = \kappa$ , it is small with respect to  $\delta$ . By the arguments of [Mag76], the cardinal  $\kappa$  becomes both the least strongly compact and the least measurable cardinal in  $V_\delta^\mathbb{P}$ . Thus, by Theorem 21, there is an inner model in which the least strongly compact cardinal is the least measurable cardinal, and by the methods from the second part of the proof of Theorem 22, this cardinal may be taken as  $\kappa$  itself.

For (2), we begin by noting that the partial ordering  $\mathbb{P}$  mentioned in the preceding paragraph is not  $<\kappa$ -friendly. However, in analogy to the first proof given for Theorem 8, for every  $\gamma < \kappa$ , let  $\mathbb{P}_\gamma$  be Magidor's notion of iterated Prikry forcing from [Mag76] which adds a Prikry sequence to every measurable cardinal in the open interval  $(\gamma, \kappa)$ . By [Mag76], forcing with  $\mathbb{P}_\gamma$  adds no subsets to  $\gamma$ . Let  $\mathbb{P}^* = \oplus\{\mathbb{P}_\gamma \mid \gamma < \kappa\}$  be their lottery sum. Magidor's arguments of [Mag76] together with  $\mathbb{P}^*$ 's definition as a lottery sum show that after forcing with  $\mathbb{P}^*$ , for some  $\gamma < \kappa$ ,  $\kappa$  is the least strongly compact cardinal, and there are no measurable cardinals in the open interval  $(\gamma, \kappa)$ . Since  $\mathbb{P}^*$  is  $<\kappa$ -friendly, (2) now follows by Theorem 10.

For (3), let  $\gamma < \kappa$ , and let  $\mathbb{P}_\gamma$  be the Easton support iteration of length  $\kappa$  which adds a non-reflecting stationary set of ordinals of cofinality  $\omega$  to every non-measurable regular limit of strong cardinals in the open interval  $(\gamma, \kappa)$ . (In other words,  $\mathbb{P}_\gamma$  does trivial forcing except at those  $\delta \in (\gamma, \kappa)$  which are non-measurable regular limits of strong cardinals, where it adds a non-reflecting stationary set of ordinals of cofinality  $\omega$  to  $\delta$ .) By the remarks in [Apt05, Section 2], after forcing with  $\mathbb{P}_\gamma$ ,  $\kappa$  becomes both the least strongly compact and least supercompact cardinal. Let  $\mathbb{P} = \oplus\{\mathbb{P}_\gamma \mid \gamma < \kappa\}$  be their lottery sum. Since  $\mathbb{P}$  is  $<\kappa$ -superfriendly, (3) now follows by Theorem 13.

Finally, for (4), we note that for any  $\gamma < \kappa$ ,  $\mathbb{P}_\gamma$  of the preceding paragraph is  $<\kappa$ -friendly, since it is  $<\kappa$ -superfriendly. Because  $|\mathbb{P}_\gamma| = \kappa$ , (4) now follows by Theorem 21 and the methods from the second part of the proof of Theorem 22.  $\square$

Before stating our next application, we briefly recall some definitions. Say that a model  $V$  of ZFC containing supercompact cardinals satisfies *level by level equivalence between strong compactness and supercompactness* if for every  $\kappa < \lambda$  regular cardinals,  $\kappa$  is  $\lambda$  strongly compact if and only if  $\kappa$  is  $\lambda$  supercompact. Say that a model  $V$  of ZFC containing supercompact cardinals satisfies *level by level inequivalence between strong compactness and supercompactness* if for every non-supercompact measurable cardinal  $\kappa$ , there is some  $\lambda > \kappa$  such that  $\kappa$  is  $\lambda$  strongly compact yet  $\kappa$  is not  $\lambda$  supercompact. Models satisfying level by level equivalence between strong compactness and supercompactness were first constructed in [AS97], and models satisfying level by level inequivalence between strong compactness and supercompactness have been constructed in [Apt02], [Apt10] and [Apt11].

**Theorem 32.**

- (1) *If the GCH holds and  $\kappa$  is  $<\delta$ -supercompact for a weakly iterable cardinal  $\delta$  above  $\kappa$ , then there is an inner model in which  $\kappa$  is supercompact and the GCH and level by level equivalence between strong compactness and supercompactness hold.*
- (2) *If the GCH holds and  $\kappa$  is  $\delta$ -supercompact for a weakly iterable cardinal  $\delta$  above  $\kappa$ , then there is an inner model in which  $\kappa$  is supercompact and the GCH and level by level inequivalence between strong compactness and supercompactness hold.*

*Proof.* For (1), assume that  $\kappa$  and  $\delta$  are least such that  $\kappa$  is  $<\delta$ -supercompact and  $\delta$  is a weakly iterable cardinal. Let  $\mathbb{P}$  be the class forcing from [AS97] defined over  $V_\delta$  such that  $V_\delta^{\mathbb{P}} \models$  “ $\kappa$  is supercompact, and the GCH and level by level equivalence between strong compactness and supercompactness hold”. We refer readers to [AS97, Section 3] for the exact definition of  $\mathbb{P}$ , which is rather complicated. We do note, however, that if  $\langle M, \delta, U \rangle$  is an iterable structure containing  $V_\delta$ , then because  $\kappa$  and  $\delta$  are least such that  $\kappa$  is  $<\delta$ -supercompact and  $\delta$  is weakly iterable,  $\mathbb{P}$  is useful for the ultrapower embedding. Therefore, by Theorem 24, there is an inner model containing a supercompact cardinal in which the GCH and level by level equivalence between strong compactness and supercompactness hold, and by the methods from the second part of the proof of Theorem 22, this cardinal may be taken as  $\kappa$  itself.

For (2), assume that  $\kappa$  and  $\delta$  are least such that  $\kappa$  is  $\delta$ -supercompact and  $\delta$  is a weakly iterable cardinal. It is a general fact that if  $\gamma$  is  $\rho$ -supercompact,  $\rho$  is weakly iterable, and  $j : V \rightarrow M$  is an elementary embedding witnessing the  $\rho$ -supercompactness of  $\gamma$ , then  $M \models$  “ $\rho$  is weakly iterable”. Therefore, by the proof of [Apt10, Theorem 2], there are cardinals  $\kappa_0 < \delta_0 < \kappa$  and a partial ordering  $\mathbb{P} \in V_{\delta_0}$  such that  $V_{\delta_0}^{\mathbb{P}} \models$  “ $\kappa_0$  is supercompact, and the GCH and level by level inequivalence between strong compactness and supercompactness hold”. Thus, by Theorem 21, there is an inner model containing a supercompact cardinal in which the GCH and level by level inequivalence between strong compactness and supercompactness hold, and by the methods from the second part of the proof of Theorem 22, this cardinal may be taken as  $\kappa$  itself.  $\square$

We mentioned at the opening of this article that we take our test questions as representative of the many more similar questions one could ask, inquiring about the existence of inner models realizing various large cardinal properties usually obtained by forcing. We would similarly like to take our answers—and in particular, the three proof methods we have described—as providing a key to answering many of them. Indeed, we encourage the reader to go ahead and formulate similar interesting questions and see if these methods are able to provide an answer. Going forward, we are especially keen to find or learn of generalizations of our first two methods, in Theorems 10, 11 and 13, which might allow us to find the more robust inner models provided by these methods for a greater variety of situations.

## REFERENCES

- [Apt02] Arthur W. Apter. On level by level equivalence and inequivalence between strong compactness and supercompactness. *Fundamenta Mathematicae*, 171(1):77–92, 2002.
- [Apt05] Arthur W. Apter. Diamond, square, and level by level equivalence. *Archive for Mathematical Logic*, 44(3):387–395, 2005.



- [Apt10] Arthur W. Apter. Tallness and level by level equivalence and inequivalence. *Mathematical Logic Quarterly*, 56(1):4–12, 2010.
- [Apt11] Arthur W. Apter. Level by level inequivalence beyond measurability. *Archive for Mathematical Logic*, 50(7-8):707–712, 2011.
- [AS97] Arthur W. Apter and Saharon Shelah. On the strong equality between supercompactness and strong compactness. *Transactions of the American Mathematical Society*, 349(1):103–128, 1997.
- [Bau84] James E. Baumgartner. Applications of the Proper Forcing Axiom. In Kenneth Kunen and Jerry Vaughan, editors, *Handbook of Set Theoretic Topology*, pages 913–959. North-Holland, Amsterdam, 1984.
- [BT09] Andrew Brooke-Taylor. Large cardinals and definable well-orders on the universe. *Journal of Symbolic Logic*, 74(2):641–654, 2009.
- [DF08] Natasha Dobrinen and Sy D. Friedman. Internal consistency and global co-stationarity of the ground model. *Journal of Symbolic Logic*, 73(2):512–521, 2008.
- [DF10] Natasha Dobrinen and Sy D. Friedman. The consistency strength of the tree property at the double successor of a measurable cardinal. *Fundamenta Mathematicae*, 208(2):123–153, 2010.
- [For09] Matthew D. Foreman. Smoke and mirrors: combinatorial properties of small cardinals equiconsistent with huge cardinals. *Advances in Mathematics*, 222(2):565–595, 2009.
- [Fri06] Sy D. Friedman. Internal consistency and the inner model hypothesis. *Bulletin of Symbolic Logic*, 12(4):591–600, 2006.
- [Fri09] Shoshana Friedman. *Aspects of HOD, Supercompactness, and Set Theoretic Geology*. PhD thesis, The Graduate Center of the City University of New York, September 2009.
- [GHJ] Victoria Gitman, Joel David Hamkins, and Thomas A. Johnstone. What is the theory ZFC without power set? Submitted for publication.
- [Git11] Victoria Gitman. Ramsey-like cardinals. *The Journal of Symbolic Logic*, 76(2):519–540, 2011.
- [GW11] Victoria Gitman and Philip D. Welch. Ramsey-like cardinals II. *The Journal of Symbolic Logic*, 76(2):541–560, 2011.
- [Ham94] Joel David Hamkins. Fragile measurability. *Journal of Symbolic Logic*, 59(1):262–282, 1994.
- [Ham00] Joel David Hamkins. The lottery preparation. *Annals of Pure and Applied Logic*, 101:103–146, 2000.
- [Ham05] Joel David Hamkins. The Ground Axiom. *Oberwolfach Report*, 55:3160–3162, 2005.
- [HS] Joel David Hamkins and Daniel Seabold. Boolean ultrapowers. In preparation.
- [Kun70] Kenneth Kunen. Some applications of iterated ultrapowers in set theory. *Annals of Mathematical Logic*, 1:179–227, 1970.
- [Lav78] Richard Laver. Making the supercompactness of  $\kappa$  indestructible under  $\kappa$ -directed closed forcing. *Israel Journal of Mathematics*, 29(4):385–388, 1978.
- [LS67] Azriel Lévy and Robert M. Solovay. Measurable cardinals and the continuum hypothesis. *Israel Journal of Mathematics*, 5:234–248, 1967.
- [Mag76] Menachem Magidor. How large is the first strongly compact cardinal? or a study on identity crises. *Annals of Mathematical Logic*, 10(1):33–57, 1976.
- [Mit79] William J. Mitchell. Ramsey cardinals and constructibility. *Journal of Symbolic Logic*, 44(2):260–266, 1979.
- [Rei06] Jonas Reitz. *The Ground Axiom*. PhD thesis, The Graduate Center of the City University of New York, September 2006.
- [Rei07] Jonas Reitz. The Ground Axiom. *Journal of Symbolic Logic*, 72(4):1299–1317, 2007.
- [Sol74] Robert M. Solovay. Strongly compact cardinals and the GCH. In Leon Henkin et. al., editor, *Proceedings of the Tarski Symposium, Proceedings Symposia Pure Mathematics, volume XXV, University of California, Berkeley, 1971*, pages 365–372. American Mathematical Society, Providence, Rhode Island, 1974.
- [Wel04] Philip Welch. On unfoldable cardinals, omega cardinals, and the beginning of the Inner Model Hierarchy. *Archive for Mathematical Logic*, 43(4):443–458, 2004.

A. W. APTER, MATHEMATICS, THE GRADUATE CENTER OF THE CITY UNIVERSITY OF NEW YORK, 365 FIFTH AVENUE, NEW YORK, NY 10016 & DEPARTMENT OF MATHEMATICS, BARUCH COLLEGE OF CUNY, ONE BERNARD BARUCH WAY, NEW YORK, NY 10010

*E-mail address:* [awapter@alum.mit.edu](mailto:awapter@alum.mit.edu), <http://faculty.baruch.cuny.edu/aapter>

V. GITMAN, MATHEMATICS, NEW YORK CITY COLLEGE OF TECHNOLOGY, 300 JAY STREET, BROOKLYN, NY 11201

*E-mail address:* [vgitman@nylogic.org](mailto:vgitman@nylogic.org), <http://websupport1.citytech.cuny.edu/faculty/vgitman>

J. D. HAMKINS, DEPARTMENT OF PHILOSOPHY, NEW YORK UNIVERSITY, 5 WASHINGTON PLACE, NEW YORK, NY 10003, & MATHEMATICS, THE GRADUATE CENTER OF THE CITY UNIVERSITY OF NEW YORK, 365 FIFTH AVENUE, NEW YORK, NY 10016 & MATHEMATICS, THE COLLEGE OF STATEN ISLAND OF CUNY, STATEN ISLAND, NY 10314

*E-mail address:* [jhamkins@gc.cuny.edu](mailto:jhamkins@gc.cuny.edu), <http://jdh.hamkins.org>